WITTGENSTEIN ON THE INFINITY OF PRIMES

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Abstract

It is controversial whether Wittgenstein's philosophy of mathematics is of critical importance for mathematical proofs, or is only concerned with the adequate philosophical interpretation of mathematics. Wittgenstein's remarks on the infinity of prime numbers provide a helpful example which will be used to clarify this question. His antiplatonistic view of mathematics contradicts the widespread understanding of proofs as logical derivations from a set of axioms or assumptions. Wittgenstein's critique of traditional proofs of the infinity of prime numbers, specifically those of Euler and Euclid, not only offers philosophical insight but also suggests substantive improvements. A careful examination of his comments leads to a deeper understanding of what proves the infinity of primes.

1 *Wittgenstein's Antiplatonism*

Throughout his life Wittgenstein contradicted the platonistic view that mathematical propositions are capable of being true or false with respect to mathematical reality. Contrary to other forms of mathematical antiplatonism, such as formalism and constructivism, Wittgenstein claims that the crucial mistake of platonism lies in interpreting mathematical propositions in terms of bivalent propositions. Propositions which are capable of being true or false in correspondence to reality are formalized in mathematical terms as functions that assign truth values to their arguments. In order to specify the extension of a propositional function, its arguments must be quantifiable. This logical formalism is common practice in proof theory and set theory. Wittgenstein objects to this practice; according to his variety of antiplatonism, mathematical propositions do not correspond to the form of propositional functions. By rejecting any logical formalization of mathematics, Wittgenstein's philosophy of mathematics stands in stark contrast to traditional approaches.

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Wittgenstein's point of view has been identified with some kinds of constructivism.[1] His criticism and transformation of Euler's proof of the infinity of primes seem to support this claim.[2] Furthermore, as with constructivism, the concept of algorithm is basic to Wittgenstein's understanding of mathematics. Contrary to constructivism, however, Wittgenstein does not argue for an alternative logical system as the basis for mathematical proofs. Rather, he calls for an understanding of mathematics that completely dispenses with logic. He not only rejects the use of certain deduction rules such as RAA (reductio ad absurdum) and DNE (double negation elimination), but also the use of any logical rules and truth functions such as negation. He does not plead for a restricted sense of logical quantification in the mathematical realm, but completely rejects the possibility of quantifying objects that satisfy propositional functions. Contrary to common constructivism, Wittgenstein's emphasis on the constructive character of mathematics leads him to abandon an understanding of mathematical proofs in terms of logical proofs in favour of a conception of mathematical proofs based on the notion of *operation*. Indeed, he places this concept in opposition to the notion of *propositional functions*. This paper demonstrates this distinction in the context of Wittgenstein's remarks on the infinity of primes.

According to *formalism*, mathematical proofs essentially rest on syntactic manipulations of signs. This is not contested by Wittgenstein. Yet Wittgenstein's philosophy runs contrary to formalism in that deductions within an axiomatic system of mathematical logic, such as Peano's or Zermelo-Fraenkel's, are not taken as paradigmatic for exact mathematical proofs. Instead, from Wittgenstein's point of view, traditional axiomatic proofs are merely an expression of misguided logical formalization. No logical formalization of mathematical propositions provable within an axiomatic system can express a purely syntactical character, because it is impossible to argue for the correctness of the axioms themselves on the basis of syntax. All that can be shown syntactically is that the theorems are derivable from the axioms – but this is not sufficient to demonstrate the truth of the theorems by purely syntactical means. Instead, according to Wittgenstein, mathematical theorems should be proven by formally expressing the mathematical proposition within an *ideal notation*, the idea being that the syntactical features of the formalization are themselves sufficient proof. Thus, a proof of the infinity of primes does not consist of a deduction from certain plain assumptions, whether formal or informal. Rather, it should be possible to construct a formal expression for the infinity of primes that proves the infinity of primes by its syntactical features. This conception of proofs will be illustrated in section 3.

[1] cf., e.g., Frascolla (1994), Rodych (1997) and Rodych (1999), Marion (1998).
[2] cf. Mancousu and Marion (2003) and section 4 of this paper.

Wittgenstein began elaborating his philosophy of mathematics early in his career, in sharp contrast to Frege's and Russell's *logicism*. In TLP 6.031, he says that the "theory of classes" at the heart of logicism is "superfluous". According to Wittgenstein, the fundamental mistake of logicism is its indifferent use of propositional functions to formalize "genuine (material) concepts" as well as "formal concepts", to define classes ("empirical totalities") and to construct systems of internally related elements such as numbers. To overcome this deficiency, Wittgenstein sharply distinguishes operations from functions and only allows mathematical propositions to be symbolized by the use of operations (WVC, p. 217, cf. p. 228, TLP 4.126-4.128, 5.25-5.252, Waismann (1982), p. 164f.):

> An operation is completely different from a function. A function cannot be its own argument. An operation, on the other hand, can be applied to its own result. In mathematics we must always be dealing with systems, and not with totalities. Russell's basic mistake consists in not having recognized the essence of a *system* while representing empirical totalities and systems by means of the same symbol – a propositional function – without drawing any distinctions.

The symbol of propositional functions, $\varphi(x)$, does not identify the objects satisfying the concept expressed by the propositional function. The extension of material concepts is thus not syntactically determined, but depends on the particular case. E.g. the concept "is a human being in this room" can be adequately formalized by a propositional function $f(x)$ because the concept is compatible with any extension. It is not the syntax of the symbols that determines which objects satisfy a propositional function, but the external properties of objects. In the case of propositional functions, truth and falsehood are thus well-defined values and it makes sense to apply the rules of first-order logic.

In contrast, in case of formal concepts the "extension" of the formal concept is syntactically determined. That is, the syntax of the symbols is sufficient to demonstrate that the symbolized objects satisfy the formal concept. In case of mathematical concepts, this syntax can be adequately expressed by symbolizing the concepts as operations rather than propositional functions (cf. TLP 4.126[4]). For example, Wittgenstein does not define the natural numbers as classes of classes (as Frege and Russell do) but as exponents of operations (TLP 6.021). Traditionally, a number such as 3 is the class of all classes having the same number of members as some representative class, which would in turn be defined by a propositional function. Instead, the number 3 can be defined by repeated application of the operation $+1$ to 0: i.e., $3 = 0+1+1+1$ (cf. TLP 6.02). This definition of the natural numbers is abbreviated by Wittgenstein as $[0, \xi, \xi + 1]$ (TLP 6.03), where the first expression defines the starting point (input) of the operation, the second stands for an arbitrary input, and the third expression defines the operation that computes the result for the arbitrary input. The result in turn defines the next input.

The formal concept "is a number" can be defined by an operation, but cannot be symbolized by a propositional function (cf. TLP 4.1272[7,8]). Consequently "3 is a natural number" is not a bivalent proposition. Due to the above definition, the number 3 can be symbolized by $0+1+1+1$; i.e., the third application of the formal concept "is a natural number" symbolized by $[0, \xi, \xi+1]$. The application of syntactical rules transforms the symbol 3 into its completely transparent form $0+1+1+1$, and the syntax of this ideal symbol is sufficient to identify 3 as a natural number. In other words, this definition of a natural number is sufficient in and of itself to construct all the objects satisfying the concept. The all-important difference between this conception and that of logicism is that the mathematical concept of a number is defined by constructing symbols with the same multiplicity as the objects they symbolize, not by propositional functions that do *not* reveal the multiplicity of the objects involved.

The rejection of propositional functions in favour of operations has far-reaching consequences. Arithmetical concepts are commonly expressed by propositional functions that are true for all numbers that satisfy the concept in question, according to the standard interpretation of some arithmetical axiomatic system. According to a standard language of mathematical logic, the following definition expresses the property of being an *even* number: $\psi(x) =_{def} \exists\nu(2 \times \nu = x)$. Wittgenstein objects to such a formalization because it represents the property of being an even number as a material and not a formal concept. According to Wittgenstein multiplication by 2 is an operation, so for any natural number it can be *syntactically* determined whether or not it is even. One simply has to symbolize the natural number by its definition in the form $0+1+1+1\ldots$, or equivalently by a succession of strokes ($|||\ldots$), then successively eliminate or bracket the pairs $1+1$ ($||$). The number in question is even iff $+1$ ($|$) does not remain after this process is carried out. While multiplication and division can be used to construct a decision rule for identifying even numbers among the natural numbers, the *general form* of even number is only adequately represented by an inductive definition. This can be given by referring to addition by 2: $[2, \xi, \xi+2]$. Similar remarks hold for other formal concepts such as being an odd number, a square number, or a Fibonacci number, all of which are better defined by operations than propositional functions. Turning to prime numbers, we will ask how they can be defined by operations and to what extent such a definition can determine the extension of the prime numbers.

According to Wittgenstein, the grammatical similarity of ordinary language expressions such as "2 is an even number" and "2 human beings are in this room" leads to the misguided view that both expressions are to be formalized by propositional functions, even though the real syntax is completely different in the two cases. Plainly, this raises a question: what are the criteria of an adequate formalization? The answer to this question, however, depends on the adequate interpretation of mathematical propositions. According to a platonistic reading, mathematical propositions are simply capable of being true or false *according to some corresponding reality*. Thus, it is not necessary to formalize them using symbolic expressions which allow us to identify their "truth" or "falsehood" on purely syntactical grounds. Wittgenstein articulates his own antiplatonistic interpretation by stipulating an alternative formal symbolization of mathematical propositions, based on the concept of operation.

In the end, the question at stake is how to interpret mathematical propositions. This question can only be discussed by looking at concrete mathematical theorems and their proofs. By discussing the theorem of the infinity of prime numbers and some of its proofs, this paper will demonstrate that Wittgenstein's understanding of mathematics is not as implausible as it might seem. Wittgenstein's distinction between propositional functions and operations in fact leads to a natural, plain, and deep understanding of mathematics. Indeed, once one gets used to it, it also culminates in a philosophically illuminating fashion of doing mathematics.

2 Ineffability of Primes

In *Philosophical Remarks*, §159[3] Wittgenstein illustrates his uncommon view that there are no open questions in mathematics. He exemplifies his view by referring to the infinity of prime numbers. Prior to any proof, most would agree that this is an open question. If one asks how many prime numbers there are, the possible answers seem pretty clear: the number of primes might be finite or infinite.

Although this approach seems evident, it is incompatible with Wittgenstein's point of view. This becomes clear if one analyzes "is a prime number" as a formal concept and applies Wittgenstein's critique of symbolizing formal concepts by propositional functions. According to Wittgenstein, questions of the form "How many x's are f?" can only be meaningful open questions when f is a material, or "genuine", concept. The question "is a human being in this room" is an example of a meaningful open question, whose answer is not syntactically determined but can only be given by referring to some external reality. This is expressed by the fact that a logical formalization of the concept "is a human being in this room" by the propositional function $f(x)$ does not allow us to derive some proposition of the form $\exists \mathrm{n} x f(x)$[4].

[3] If no other source is given, all quotations in this section are taken from this paragraph.

[4] This is shorthand notation for $\exists x_1 \ldots \exists x_n (f(x_1) \wedge \ldots \wedge f(x_n))$, whereby different variables have to be satisfied by different names denoting different objects.

The "concept prime number", Wittgenstein says, "is given to me in a completely different way from a genuine concept". He starts to illuminate this idea by referring to the "strict expression of the proposition '7 is a prime number"':

> For what is the strict expression of the proposition '7 is a prime number'? Obviously it is only that dividing 7 by a smaller number always leaves a remainder. There cannot be a different expression for that, since we can't describe mathematics, we can only do it. (And that of itself abolishes every 'set theory'.)

"7 is a prime number" is a proposition of ordinary language with the same grammatical form as a meaningful, bivalent proposition. Wittgenstein confronts this misleading ordinary language expression with the idea of a "strict expression", i.e., an adequate formalization within an ideal notation. Contrary to ordinary language, Wittgenstein's adequate formal expression renders transparent the fact that 7 is a prime number through its very syntax. Substituting ||||||| for 7, we can represent division of 7 by the numbers n $(1 < n < 7)$ by grouping the sequence of strokes into sequences of n strokes. It then suffices to determine in every case that some strokes remain: (||)(||)(||) |, (|||)(|||) |, (||||) |||, (|||||) ||, (||||||) |. This representation satisfies Wittgenstein's criterion for a formal concept. To identify the natural numbers satisfying a formal concept, it is sufficient to apply the formal operations that define the concept to the sequences of strokes defining the natural numbers. Thus, the concept of a prime number must not be symbolized by a propositional function, but by an operation applied to symbols of numbers. Most interestingly, Wittgenstein justifies this view in terms of his antiplatonistic understanding of mathematics, which he opposes to "set theory". Therefore, it is evident that an adequate formal representation of a prime number depends on one's interpretation (platonistic or antiplatonistic) of mathematical propositions.[5]

Wittgenstein's formal expression of the prime numbers is incompatible with their common formalization within mathematical logic. Prime numbers are commonly defined by the propositional function $\chi(x) =_{def} (x \neq 1 \wedge \forall u \forall v (u \times v = x \rightarrow (u = 1 \vee v = 1)))$.[6] According to Wittgenstein's view, this is an example of "the curse of the invasion of mathematics by mathematical logic" using "a method of writing [that] is nothing but the translation of vague ordinary prose." (RFM, V, §46) In Wittgenstein's analysis, an adequate formalism does not symbolize formal

[5] Wittgenstein also refers to his antiplatonism in his discussion of the infinity of prime numbers in PR, §157, by saying that "in mathematics, the signs themselves *do* mathematics, they don't describe it".

[6] This definition is primitive recursive, cf. Odifreddi (1989), p. 25. Wittgenstein's rejection of it shows that his conception of an operation is not identical with the ordinary conception of recursion, cf. Frascolla (1994) and Marion (1998).

concepts by propositional functions but by variables for which rules of substitution are defined (cf. TLP 4.1272[7,8]). Wittgenstein, for example, identifies an inductive rule of substitution that allows one to construct all the natural numbers (cf. WVC, p. 109). In other words, $[0, \xi, \xi + 1]$ defines not only the natural numbers but also the rule of substitution for a variable of natural numbers. Wittgenstein calls his inductive rule "the general form of natural numbers" and identifies it with the "concept of a natural number" (TLP 6.022[1], 6.03). This conception can be applied to even, square, and Fibonacci numbers, all of which can also be defined inductively. There is a crucial difference in the primes, however: no inductive definition of the prime numbers is yet available. That is to say, there is no rule that allows to compute the n+1th prime number from the given first n prime numbers. Instead, a definition of primes in the form of a decision rule can be given. The following rule is actually applied by Wittgenstein in the passage quoted above:

DRPRIMES: A natural number x is prime iff dividing x by the natural numbers i ($1 < i < x$) always yields some natural number y and a remainder r ($r > 0$).

Here the only arithmetical operation referred to is division, whose definition can be traced back to syntactic manipulations of number symbols as we have already seen. According to Wittgenstein, an identity in mathematical equations cannot be expressed as a propositional function, but only as a formal relation between the results of applying operations on both sides of the identity sign. The variables x, i, y, r are not bound by quantifiers, because they do not occur in propositional functions – they are just placeholders for the basis and results of operations. To this extent, defining DRPRIMES satisfies the criteria imposed by Wittgenstein on the possible definitions of formal concepts. In fact, Wittgenstein calls the "concept of a prime number" "the general rule of deciding whether a number is prime or not" or equivalently "the general form of investigating a number concerning the corresponding property" (PR, §161). However, the "concept of 'prime number"' cannot be identified with "the general form of prime numbers" that Wittgenstein refers to in the third paragraph of PR, §159:

> Therefore once I can write down the general form of prime numbers, i.e., an expression in which anything analogous to 'the number of prime numbers' is contained at all, then there is no longer a question of 'how many' primes there are. Until I can do this, I can't even pose the question. For I can't ask 'Does the series of primes *eventually* come to an end?' or 'Does another prime *ever* come after 7?'

Thus, "the general form of prime numbers" cannot be identified with the decision rule DRPRIMES but has to be defined with an inductive definition of the primes for the following reasons.

(i) The formulation of the first sentence – “Therefore once I can write down the general form of prime numbers . . . ” – suggests that the general form of primes is still to be defined. This makes sense, assuming that “the general form of prime numbers” refers to an inductive definition, because finding such a definition is a crucial and hitherto unsolved “problem” of the theory of primes.

(ii) Wittgenstein uses the phrase “the general form of prime numbers” in a manner similar to “the general form of an integer” in TLP 6.03, and the latter is identified with an inductive definition.

(iii) According to Wittgenstein, the variable for a certain kind of number is defined by the general form of that kind, and only an inductive substitution rule is able to serve for these variables (cf. WVC, p. 82, 109). The decision rule for primes cannot serve as a substitution rule for a variable of primes, because it is not inductive. Thus, DRPRIMES is not identical with the general form of primes.

(iv) If the general form of primes is an inductive definition, the sense of the expression “anything analogous to ‘the number of prime numbers”’ becomes clear: Wittgenstein persistently emphasizes that the only way to sensibly speak of the infinite is by determining an applicable iterative operation such as an inductive rule (cf. WVC, p. 227-231). He thereby rejects any “actual”, extensional definition of infinity. Strictly speaking, it does not make sense to speak of infinite primes in the same context that one speaks of three or a hundred primes, because “infinity” is not a number. Rather, it is the possibility of constructing more numbers by iteratively applying some operation. Unlike the decision rule for primes, an inductive definition of the prime numbers would satisfy this condition.

(v) If the primes are defined inductively, it also becomes clear why it does not make sense to ask about the number of primes: once an inductive rule is given, it is senseless to ask about the greatest prime (WVC, p. 135f.).

(vi) Distinguishing “the general form of prime numbers” from the concept of a prime also explains why Wittgenstein says that the question of how many primes there are cannot be posed beforehand: he refers to the situation where a decision rule is available but no inductive rule. This is supported by a similar passage from WVC, p. 136:

> Within *that* system in which I come to know that a certain number is prime I cannot even ask what the number of primes is. [. . .] And once you

> have discovered the induction, this is again a different matter from computing a certain number.

Wittgenstein also distinguishes between an inductive rule to construct the primes and the rule to identify primes in PR, §190[1]. Only the former allows one to perceive a law in the distribution of primes; the latter merely identifies single primes:

> The primes likewise come from one's method of looking for them, as the result of an experiment. [. . .] I have only found the number, not generated it. I look for it, but I don't generate it. I can certainly see a law in the rule which tells me how to find the primes, but not in the numbers that result. And so it is unlike the case $+\frac{1}{1!}, -\frac{1}{3!}, +\frac{1}{5!}$ etc., where I can see a law *in the numbers*. I must be able to write down a part of the series, in such a way that you can *recognize* the law.

According to Wittgenstein's standards, a properly defined inductive rule for the primes must make it possible to construct the series of primes *without applying DRPRIMES*. Checking the natural numbers one by one in order to find out whether or not they are primes is insufficient.

In the last sentence of PR, §159[3] (quoted above), Wittgenstein explains why the decision rule for identifying prime numbers does not allow one to pose a question regarding the number of primes: it does not contain any information about the enumeration of primes or a rule for generating successive primes – it only allows us to enumerate the primes by investigating the natural numbers one by one. DRPRIMES neither excludes the possibility of a greatest prime nor does it offer any criterion to look for a greatest prime – contrary to an inductive rule DRPRIMES does not guarantee the possibility to construct an endless series of primes. One also has to bear in mind that Wittgenstein rejects any symbolization of primes by propositional functions. Consequently, he rejects questions that require answers of the form $\exists\text{n}x f(x) \wedge \neg\exists\text{n+1}x f(x)$ or $\exists x(x > 7 \wedge f(x))$. Thus, neither an adequate formal representation of the concept of primes nor the general form of primes will allow one to ask about the number of primes.

However, Wittgenstein's reasoning is limited to the context of an exact formal language, which in his view adequately formalizes the concept of prime numbers. Thus, he only shows that it is impossible to ask about the extension of prime numbers within this context. It is of course possible to ask "How many prime numbers are there?" in ordinary language. Without proof one does not know the answer, so the proof seems to answer an open question. Wittgenstein concludes PR, §159 by analyzing this situation:

> For since it was possible for us to have the phrase 'prime number' in ordinary language, even before there was a strict expression which so to speak admitted of having such a number assigned to it, it was also possible for people to have wrongly formed the question of how many primes there were. This is what creates the impression that previously there was a problem, which is now solved. Verbal language seems to permit this question both before and after, and so creates the illusion that there had been a genuine problem which was succeeded by a genuine solution. Whereas in exact language, people originally had nothing of which they could ask how many, and later had an expression from which one could immediately read off its multiplicity.
>
> Thus I want to say: only in our verbal language (which in this case leads to a misunderstanding of logical form) are there 'as yet unsolved problems' in mathematics, or problems of the finite 'solubility of every mathematical problem'.

According to this analysis, the misleading grammar of ordinary language is responsible for raising questions as to the number of primes. We use the word "prime number" in ordinary language like any other grammatical predicate, and feel free to say that a "natural number is a prime number iff it cannot be divided by any natural number except itself and 1". In such a statement the subject of interest seems to be clear and the question of numbering the primes seems sensible. Contrary to the syntax of an exact, adequate formalism, the syntax of ordinary language makes no difference between formal concepts and material concepts. It does not show us that a question regarding the number of primes is different from a question regarding the number of humans in a room.

Wittgenstein neither rejects the possibility of posing the question in ordinary language, nor neglects the situation before the infinity of primes is proven. Neither does he deny that the proof has some connection to the question: one stops asking the question after a proof is available. What he denies is that the original question was as well defined as a question of how many objects satisfy some material concept. In the case of primes, we possess the criterion DRPRIMES to identify them one by one but no inductive rule to construct them. Consequently, DRPRIMES does not permit us to identify the extension of this formal construct and we have no way of answering questions about their number.

In this situation, one can neither "seek" for a solution nor even be sure that some solution is available (cf. WVC, p. 136). Thus, the infinity of primes is not a "genuine problem" with a "genuine solution". Any "solution" consists only of stipulating criteria which enable us to identify the extension of primes. In the case of formal concepts, such criteria are not independent of their extension. They themselves have the form of operations, in that they define a method to construct objects with the property in question. Of course, the definition of such an operation is not arbitrary – it has to be compatible with the known definition of a prime (e.g., DRPRIMES).

One might try to come up with an inductive definition, e.g. by investigating finite sequences of primes and trying to identify some arithmetical rule allowing the series to be constructed. Yet no method has been defined that could lead to such an operation in question – we must first establish the connection between the existence of a series of primes identified by applying the decision rule to natural numbers and the existence of a general rule that allows one to construct the prime numbers. In terms of the "unanswered questions of mathematics", one does not really know what one is asking unless one already has the answer. The proper approach is to stop asking vague questions, and focus on clarifying the relevant concept.

In the cited passage Wittgenstein no longer speaks of a "strict expression" of prime numbers as if it were something the future *might* reveal by an inductive definition. He describes it as something whose existence is unquestionable, since a proof of the infinity of primes is known. He says that the problem of identifying the number of primes "is now solved", and mentions that (while maintaining that originally it was not possible to ask about their number) an expression is now available that allows one to immediately read off their multiplicity. Similarly, he notes on other occasions that it has been proven by induction that the number of primes is infinite. He also mentions that as a consequence of the proof, human beings stopped looking for the greatest prime number (e.g. WVC, p. 136). However, it remains unclear whether the infinity of primes has in fact been proven by induction. The most famous proof of Euclid is usually presented as a reductio ad absurdum, and the well-known and influential proof of Euler does not even define a rule allowing the construction of prime numbers. By examining these proofs in the following sections, I shall elaborate how Wittgenstein's point of view applies to these proofs. In doing so, the "strict expression" that identifies the number of primes will be made explicit. I will also explain how it is possible to define an operation that can be used to construct some primes without defining the general form of primes. In the process, I shall also define more precisely what it means "to construct numbers" and give a constructive proof satisfying Wittgenstein's criteria.

3 Critique and Transformation of Euclid's Proof

Euclid's proof of the infinitude of primes is usually reconstructed as an informal, indirect proof with the following assumptions:

Assumption 1: $p_1 \ldots p_n$ are primes.

Assumption 2: $p_1 \ldots p_n$ are *all* primes.

Assumption 3: $k = \prod_{i=1}^{i=n} p_i + 1$ is not divisible by $p_1 \ldots p_n$.

Assumption 4: k is divisible by at least one prime number.

From assumptions 1, 2 and 4 it follows that k must be divisible by at least one of the primes $p_1 \ldots p_n$. This contradicts assumption 3. By RAA, assumption 2 (the only assumption not based on fundamental arithmetical laws) is negated; thus, the set $p_1 \ldots p_n$ does not include all primes. From this result and assumption 1, the infinity of primes follows.

Theorem: n primes exist, and if n primes exist then $n + 1$ primes exist.

The series $p_1 \ldots p_n$ does not have to be identical with a monotonically increasing series of primes, nor does it have to contain all primes within its range.[7] $p_1 \ldots p_n$ simply represents some finite enumeration of primes. p_1 does not need to be 2, the smallest prime, and p_n does not need to be the "largest prime" by hypothesis. For instance, p_1 might be 3 and $p_2 = p_n = 5$: In this case $k = 16$, and $p_{n+1} = 2$. Thus, the proof does not imply that $p_{n+1} > p_n$ as is often presumed.[8]

Euclid's proof is often quoted as a paradigm of a conclusive, unquestionable mathematical proof. In its usual form, however, it does not satisfy the criteria that Wittgenstein imposes on adequate mathematical proofs. To see this, recall that Wittgenstein excludes any informal proofs which are derived from accepted assumptions by the logical rules of natural deduction. This will be shown below by applying Wittgenstein's antiplatonistic criticism to Euclid's proof. At first glance the criticism seems incompatible with such a fundamental mathematical proof, and his stance seems overly radical. However, we go on to show how Euclid's proof can be transformed to satisfy Wittgenstein's conception of a mathematical proof. By doing this, we intend to demonstrate that Wittgenstein's antiplatonism does not lead to a rejection but to an improved understanding of Euclid's proof. Before we will elucidate the motivation of Wittgenstein's criticism.

The problem with resting on accepted assumptions is that such informal proofs rely on subjective evidence. According to the familiar conception, a mathematical proof is a logical deduction based on assumptions that in turn rest on axioms which cannot be argued for, but must simply be accepted as evident. Wittgenstein demands that all references to subjective evidence have to be eliminated. In

[7] cf. Hasse (1950), p. 4f.

[8] e.g. Dickson (1952), p. 413, Mancousu and Marion (2003), p. 185, footnote 8 and Marion (1998), p. 86.

a mathematical proof it must be possible to answer the question "Why does this theorem hold?" by referring to some syntactical feature of the formal expressions. Typically, a proof reveals such features by applying certain mechanical rules of manipulating symbols, as in the solution of equations.

According to a current understanding of a mathematical proof, however, the answer to the question "Why does this theorem hold?" is of the form "Because it follows from propositions accepted as evidently true". This clearly does not answer the question by referring to objective, syntactical criteria. At best, this approach can answer the question of why one *believes* in the validity of some theorem. This can be contrasted with an arithmetical calculation, e.g., which proves an equation independently of anyone's beliefs. According to Wittgenstein's conception, all the efforts made to convince people of a proof are not in fact part of the proof, but merely part of its accompanying prose. As we will see, according to Wittgenstein's analysis the Euclidian proof of the infinity of primes suffers from a confusion of proof and prose.

A subjective understanding of mathematical proofs is just the reverse of platonism. When a reference to non-syntactical features is necessary to ensure the truth of a mathematical proposition or axiom, then the truth of this basic proposition *must* be accepted as evident if it is to be used at all. As explained in section 1, these references to non-syntactical features essentially stem from the logical interpretation of mathematical propositions treated as propositional functions. In Euclid's proof, this becomes clear when formalizing the informal argument:

$$\bigwedge_{i=1}^{i=n} P(p_i),\ \ \neg\exists \text{n+1} x P(x),\ \ \bigwedge_{i=1}^{i=n} \neg D(p_i),\ \ \exists x (P(x) \wedge D(x)) \vdash$$

$$\exists \text{n} x P(x) \wedge (\exists \text{n} x P(x) \rightarrow \exists \text{n+1} x P(x))$$

Key:

$P(\_)$: _ is prime, $D(\_)$: $k = \prod_{i=1}^{i=n} p_i + 1$ is divisible by _.

From assumptions 1, 2, and 4 $\bigvee_{i=1}^{i=n} D(p_i)$ follows. From this, by definition of the disjunctor, $\neg \bigwedge_{i=1}^{i=n} \neg D(p_i)$ can be derived. This contradicts assumption 3. Thus by RAA the negation of assumption 2 follows, and by DNE we can derive that $\exists \text{n+1} x P(x)$. From assumption 1 $\exists \text{n} x P(x)$ follows, and by the rule of conditional proof $\exists \text{n} x P(x) \rightarrow \exists \text{n+1} x P(x)$ can be concluded. Thus, by $\wedge$-introduction the conclusion is derivable.

The formalization makes transparent the presumed logical reading of the mathematical propositions involved, based on the *propositional functions* $P(\_)$ and

$D(\_)$. Although the assumptions might themselves be the object of further proofs, at some point one will arrive at axioms that stipulate the existence of certain objects or the generality of certain concepts without any support from syntactic criteria. Such a proof is unable to explain why the theorem holds by referring to objective criteria. The aim of Wittgenstein's mathematics is to overcome this deficiency by accepting only *syntactical criteria* as valid reasons for mathematical theorems. With respect to the foundations of mathematics, this is an acceptable motivation. There are only two types of mathematical proofs that satisfy Wittgenstein's conception, WVC, p. 135:[9]

> In mathematics there are *two kinds of proof*:
>
> 1. A proof proving a particular formula. This formula occurs in the proof itself, as its last step.
>
> 2. Proof by induction. Here it is first of all striking that the proposition to be proved does not occur in the proof itself at all. Thus the proof does not actually prove the proposition. That is to say, induction is not a procedure leading to a proposition.

Proofs by induction put an end to puzzling and ill-defined questions such as the number of primes (cf. the analysis on p. 10). The first category includes those ordinary proofs of well-defined problems commonly given as "pieces of homework" (cf. PR, §158) to mathematics students. According to Wittgenstein, proofs by induction are not of type 1; he maintains that an inductive rule is not a procedure that proves a theorem, but an expression of the theorem itself. This interpretation of induction will be clarified in relation to the Euclidian proof below. Wittgenstein sometimes reserves the term "proof" exclusively for proofs of the first kind (cf. WVC, p. 33). This is due to his demand that it must be possible to distinguish the theorem to be proven from the application of some procedure to prove it. However, these questions of terminology are not the most important point. Wittgenstein does not want to deny that inductive rules play an important role in the progress of mathematics. Instead, he is stressing their fundamental significance for mathematics. Generalizing the usual conception of an inductive proof and referring to Wittgenstein's basic concept of operation, one might distinguish the two kinds of proof in the following way. Proofs of the first kind are mechanical applications of previously defined operations with a finite number of steps, whereas a proof of the second kind consists of a definition of operations. As Wittgenstein's mathematics basically consists of defining and applying operations, these two proofs suffice to specify mathematical proofs deserving of the name. Before rejecting Wittgenstein's conception of mathematical proofs as overly narrow, one should consider

[9] Among proofs of type 1, Wittgenstein considers arithmetical or algebraic equations and not logical deductions, cf. his examples in WVC, p. 33 and 135.

its consequences in detail. In WVC, p. 136, Wittgenstein explicitly mentions that one "proves by induction that an infinite number of primes exist." In the following discussion, I will explain how the Euclidian proof can be transformed to become subsumable under the concept "a proof by induction". Furthermore, it will be argued that this transformation improves the proof.

As long as one does *not* interpret Euclid's proof as a proof by induction, it suffers from another deficiency. Wittgenstein mentions this problem in reference to Euler's proof (WVC, p. 108f.), but it remains just as valid for Euclid's.

> Euler's proof is immediately in error, as soon as prime numbers are written down in the form $p_1, p_2 \ldots p_n$. For if the index n is to mean an *arbitrary* number, then this already presupposes a law of progression, and this law can be given only in terms of an induction. Thus the proof presupposes what it is supposed to prove.

As its first assumption makes evident, the common form of Euclid's proof suffers from the same deficiency: prime numbers are here written down in the form $p_1, p_2 \ldots p_n$ and p_n denotes an *arbitrary* prime number. n is a variable that might take any natural number as its value. p_n thus can only denote a prime number if any number of primes, i.e., an infinite number of primes exists (cf. WVC, p. 109). Thus the proof suffers from petitio principii: assumption 1 presupposes the infinity of primes in order to be meaningful. The only way to avoid the petitio is to define an inductive rule that constructs an arbitrary number of primes (cf. PR, §129). The definition of such a rule is both what the proof is asking about and what it consists of. This is actually how Wittgenstein interprets the Euclidian proof; for this reason he only criticizes Euler's proof.

According to Wittgenstein, Euclid's proof consists of nothing but the definition of an inductive rule for constructing an infinite series of primes from an arbitrarily chosen first prime. If one takes 2 as starting point, the inductive rule can be expressed as

$$[2, p_1 \ldots p_n, p_{n+1} = j_s], \tag{1}$$

where j_s is the smallest j satisfying the following equation (y refers to a natural number):[10]

---

[10] Instead of $1 < j \leq \prod_{i=1}^{i=n} p_i + 1$, it would suffice to let j go from 2 to $\sqrt{\prod_{i=1}^{i=n} p_i + 1}$. If no j satisfies the equation mentioned in the text, then $j_s = \prod_{i=1}^{i=n} p_i + 1$. For the sake of a simple expression (as opposed to a simple computation), we will abstain from this rule.

$$\frac{\prod_{i=1}^{i=n} p_i+1}{j} = y \ \ (1 < j \leq \prod_{i=1}^{i=n} p_i + 1). \tag{2}$$

By applying (1) iteratively, we can construct the series of primes starting with 2,3,7,43,13,53,5 . . . .

According to Wittgenstein, the adequate formal expression of the infinity of primes is not the logical formalization $\exists \text{n} x P(x) \wedge (\exists \text{n} x P(x) \rightarrow \exists \text{n+1} x P(x))$, but the inductive definition given in (1) and (2). As Wittgenstein repeatedly states, the definition of an inductive rule is nothing more than an *expression* demonstrating the existence of an infinite series of numbers of a certain kind. It is not a method for drawing conclusions about further propositions, which can only state what the definition already expresses (cf. e.g. WVC, p. 33, 82, 135, PR, §126, 129, PR, II, VI). Thus, (1) and (2) are expressions of the proven theorem and not a method of deriving the theorem as a logical proposition. Euclid's proof, as interpreted above, is an strict expression that proves the infinity of primes simply by virtue of inductively generating primes. That each application generates a new prime is proven by the *syntactical feature* $+1$ in the numerator of (2). If we define each number as a series of strokes, then the numerator consists of a number of strokes that by definition (see our previous definition of the division operator) is *not* evenly divisible by $p_1 \ldots p_n$. Thus, the construction of primes is wholly syntactically determined and independent of any reasoning or evidence not delivered by the symbols themselves.

One might object to (1) and (2) on the basis that these equations are simply a succession of signs, and do not prove anything unless one understands them and comes to believe that by applying the inductive definition an infinite series of primes is generated. Wittgenstein's point of view is that the *signs* are not sufficient; it is the *symbols* (i.e., the signs with their syntactical rules) that prove a mathematical theorem. Thus, (1) and (2) should be read not as strings but as an inductive definition of an infinite series of primes. Of course explanations are needed to understand (1) and (2) and become convinced that these expressions define an infinite number of primes. These explanations are part of the prose of the proof – they do not prove the theorem, but explain it. The inductive definition is sufficient to generate any number of primes by computer, yet might not be enough to convince humans that an infinite number of primes can be computed by applying the rule. In the traditional form of Euclid's proof, prose and proof are confused. As their logical formalization shows, its assumptions do not contain syntactical features that prove what is in question. Instead they state what should be syntactically proven, in the form of mathematical propositions capable of being true and false.

For instance, assumption 4 states (by material implication) that the product of n primes plus 1 will not be divisible by one of those n primes, even though this must follow from the definition of these numbers and arithmetical operations. Ordinary language sentences such as assumption 4 can be used to call someone's attention to certain arithmetical relations, but this is a way of explaining the proof and does not actually mention a part of it. Thus, so-called informal proofs are not genuine mathematical proofs, but simply their ordinary language explanations. Such proofs might serve to make mathematical theorems more palatable, but they also obscure the exact form of the proofs.

Definition (1) is inductive in the sense that its inputs are also outputs of previous applications of the operation. Yet this is not necessarily the case when it comes to defining an infinite series of primes, as the following variation of (1) and (2) demonstrate. By simply substituting i for p_i in formula (2), one can define an operation that constructs a series of primes by computing the ith member of the series starting from 1. p_i is then the smallest j (i.e., j_s) satisfying the equation

$$\frac{\prod_{i=1}^{i=n} i+1}{j} = y \ \ (1 < j \leq \prod_{i=1}^{i=n} i + 1). \tag{3}$$

Unlike (1), this rule defines a series of primes where the result of each operation is not identical with the next input. Moreover, two operations are involved: one that defines how to compute a prime given a natural number i, and one that defines the series of natural numbers used as inputs. This might be symbolized by

$$[1, i, {}^{p_i = j_s}_{i+1}]. \tag{4}$$

The upper expression in the third term defines the operation that computes a result for input i, while the lower expression defines the operation that computes the next input. Iterative application of (4) yields the series $2, 3, 7, 5, 11, 7, 71, 61, 19, 11, 39916801, \ldots$.

Note that the rule for computing the series of primes according to definition (4) does not guarantee that p_i and p_{i+j} are not identical; the defined series might contain redundancies. The third and sixth prime number of the series are both 7, for example. This is a disadvantage compared to definition (1). The series of primes so defined is also not monotonically increasing, but despite these facts the rule still ensures that every p_k is eventually succeeded by a prime $> p_k$. This follows because at some point $i = p_k$, and the decomposition of $\prod_{i=1}^{i=p_k} i + 1$ into primes will lead to a prime $> p_k$.

Definition (4) is actually preferable to (1) in some respects.

(i) The ith member of the resulting series can be directly computed by taking the natural number i as input whereas definition (1) requires the computation of all preceding members in the series.

(ii) Rule (4) ensures that each p_i will lie between i and $i! + 1$. In the interval j to $i! + 1$ we might find that p_j $(i < j \leq i! + 1)$ is less than p_i, but the series is essentially increasing if one considers $j > i! + 1$. In contrast, one cannot define some p_j in the series of rule (1) that is always greater than some p_i $(i < j)$.

(iii) From this it follows that definition (4) allows us to solve some problems that (1) is unable to solve. Given some number i, we can define an interval that surely contains a prime $> i$. The solution is given by the rule $i < p_i \leq i!+1$.

(iv) Finally, definition (4) demonstrates that it is possible to prove the infinity of primes without referring to $p_1 \ldots p_n$. It is remarkable that even though many proofs of the infinity of primes have been given, several being variants of Euclid's proof, nobody has ever proposed a definition like (3) or something analogous that does *not* depend on the assumption $p_1 \ldots p_n$.[11]

The relation between (1) and (4) is similar to the relation between the inductive definition of the Fibonacci numbers and Binet's formula, which allows one to compute the n-th Fibonacci number given n. However, both the inductive rule and Binet's formula define a monotonically increasing series of *the* (i.e., all) Fibonacci numbers. In contrast, (1) and (4) both suffer from the crucial deficiency that the i-th prime in the series is not identical with the i-th prime in the complete series of monotonically increasing primes. (The series of primes yielded by applying DRPRIMES to the series of natural numbers.) Thus, there is no guarantee that all the primes will appear in the series defined by (1) and (4), respectively. That is why neither of these rules is the general form of the primes. It is thus quite possible to prove the infinity of primes without defining the general form of the primes, because it is possible to define operations constructing other infinite series of primes.

This is due to the fact that both (1) and (4) still refer to the decomposition of numbers into prime numbers. They do not make DRPRIMES superfluous by defining arithmetical rules that would directly compute a prime number. Instead, both (1) and (4) are forced to compute numbers that are not necessarily primes. These operations always allow us to identify a new prime in a finite number of steps, but they do not directly compute prime pairs such that between p_i and p_{i+1} $(p_{i+1} > p_i)$ there is no natural number that is a prime. Yet this is just what would

[11] For an overview of proofs of the infinity of primes, cf. Dickson (1952), p. 413.

be required of a definition of the *general form* of the primes. Only such a definition would reveal the essential nature of primes.

Wittgenstein's notion of an "inductive proof" should be taken in the broader sense, that of an operation which constructs a series of numbers with a certain property. The resulting series need not contain all the numbers of that property to satisfy the concept of an inductive proof. The inductive proof essentially provides a rule to construct a further member of a series. This rule may well construct it from previous construed members as in case of (1) or simply construct the ith member given i as in case of (4). "To construct" numbers means to iteratively apply some defined operation or composition of operations and thereby obtain a series of numbers. The notion of a "constructive proof" in the sense of Wittgenstein thus has to be defined using the basic concept of an operation: adequate mathematical concepts either consist of a definition of operations, or an application of defined operations. In the first sense the proof is of type 2, in the second sense the proof is of type 1 (according to WVC, p. 135, quoted above on p. 14). This is the sense in which Wittgenstein's conception of a mathematical proof is "constructive", and rejects any non-constructive proof. By discussing Euler's proof, we will now see how Wittgenstein deals with a proof that is essentially "non-constructive". Unlike the Euclidian proof, Euler's proof does not involve any rule for constructing an infinite series of primes.

4 Critique and Transformation of Euler's Proof

Euler claims to have proven the infinity of primes by deducing it from the equation

$$\sum_{m=1}^{\infty} \frac{1}{m} = \prod_{i=1}^{n} \frac{1}{1-\frac{1}{p_i}} \tag{5}$$

n being the number of primes. Equation (5) is actually invalid, as Kronecker pointed out. Kronecker rectifies this deficiency by adding the exponent z ($z > 1$) to m and p_i (cf. Mancousu and Marion (2003), p. 173f.). In addition to (5), Euler presumes that the sum of the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ on the left-hand side is infinite, as he had shown in another proof. Euler proves the infinity of primes by showing that the right-hand side is finite as long as n is a finite natural number. Thus, i has to go from 1 to ∞: there is no last prime number.

The inequality of the left- and right-hand sides of (5) in the case of finite n can be made transparent by the following argument: In the case n is some finite number k, the right-hand side of (5) is equal to

$$(1 + \tfrac{1}{2} + \tfrac{1}{2^2} + \ldots) \cdot (1 + \tfrac{1}{3} + \tfrac{1}{3^2} + \ldots) \cdot \ldots \cdot (1 + \tfrac{1}{p_k} + \tfrac{1}{p_k^2} + \ldots) \tag{6}$$

This is in turn equal to

$$\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \ldots \cdot \frac{p_k}{p_k - 1} = \prod_{i=1}^{i=k} \frac{p_i}{p_i - 1} \tag{7}$$

This again is equal to 1 plus an infinite number of summands $\frac{1}{i}$, where i represents all the numbers that can be factored into the primes $p_1, p_2, \ldots p_k$. In this sense the right-hand side of (5) defines an infinite series even when n is a finite number k. In the case $k = 1$, for example, the series contains only all even numbers in the denominators, i.e., all and only numbers that can be factored by the first prime number: $1 + \frac{1}{2} + \frac{1}{4} + \ldots$. This series is not identical with the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ defined by the left-hand side of (5), because it does not contain the summands with uneven numbers in the denominator. This result can be generalized as follows. An infinite series $1 + \frac{1}{2} + \ldots$ containing all and only those fractions whose denominators can be factored by a finite set of primes is not identical to the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$. There are always some terms in the latter which are not found in the former, so long as n is finite. However, one cannot tell from the proof which ones are missing.

This does not satisfy Wittgenstein's conception of a mathematical proof, for the following reasons. (i) It neither applies nor defines any operation to construct primes. Even given a finite number of primes, the proof does not provide any method of deriving a *new* prime from the infinite series defined by the left-hand side of (5). Instead, the argument relies only on the inequality of the limits in equation (5) in case n is finite (cf. WA2, p. 321-325, PG, II, V, 26, WA11, 123). (ii) It makes use of the natural variable n in order to denote some arbitrary prime number p_n. Thus, it presumes the existence of any arbitrary number n of primes, and thus the existence of an inductive rule to construct an infinite series of primes. It is equivalent to the use of $p_1 \ldots p_n$ by Euclid, which presupposes what is in question (cf. WVC, p. 108 and p. 15 above).[12]

Although Wittgenstein criticizes Euler's proof, he does not abandon it completely. Rather, his criticism is that it does not adequately express the proving element. The problem with Euler's proof is this: it seems to state that $\sum_{m=1}^{\infty} \frac{1}{m}$ is infinite, and thus larger than any finite product $\prod_{i=1}^{i=k} \frac{1}{1 - \frac{1}{p_i}}$ with k being a natural number. Yet according to Wittgenstein, infinity is not itself a number or extension that can be meaningfully compared to some material number. If one wishes to compare two series, then a method of comparison has to be defined. In the case

[12] Mancousu and Marion (2003) do not recognize this criticism, cf. their footnote 18.

of infinite series, some operation which correlates the elements of the two series is needed. This is lacking in Euler's proof.

Correctly analyzing what is meant when Euler states that $\sum\limits_{m=1}^{\infty} \frac{1}{m}$ is "infinite", however, leads to an equivalent constructive proof. According to Wittgenstein, the divergence or infinity of $\sum\limits_{m=1}^{\infty} \frac{1}{m}$ implies that a finite, specifiable number of summands from this series will always exceed a certain number. His transformation of Euler's proof consists of defining an operation that calculates these summands. Given some number n, he proves that $\sum\limits_{i=n}^{i=3n-1} \frac{1}{i} \geq 1$.[13] From this it follows that $\sum\limits_{i=1}^{i=4^m} \frac{1}{i} > m$. According to Wittgenstein, *this* is the adequate expression for the infinitude of the harmonic series $\sum\limits_{m=1}^{\infty} \frac{1}{m}$. It defines a rule depending on an arbitrary number m for constructing a sum that is larger than m.

Up to this point there has been a close analogy between Wittgenstein's transformation of Euler's proof and the transformation carried out by Behmann:[14] Behmann also bases his proof on a rule for constructing a sum that is larger than an arbitrary real number r. His calculations even define a smaller upper bound than Wittgenstein's, namely 2^{2r-1} (i.e., just half of Wittgenstein's 4^m as Mancousu and Marion (2003), p. 179 point out). When it comes to applying their refinements of the left-hand side of (5) to Euler's proof, however, Behmann and Wittgenstein differ significantly. This will be shown in the following discussion.[15]

Wittgenstein concludes his discussion of Euler's proof by applying his analysis of the infinity of the harmonic series to the proof of the infinitude of primes (WA2, p. 325, translation taken form Mancousu and Marion (2003), p. 179):

> The sum of the series $1 + \frac{1}{2} + \frac{1}{3} + \ldots$ developed up to 4^m thus surely exceeds m. Thus, we have

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{4^m} > (1 + \frac{1}{2} + \frac{1}{2^2} + \ldots) \cdot (1 + \frac{1}{3} + \frac{1}{3^2} + \ldots) \cdot \ldots \cdot (1 + \frac{1}{m} + \frac{1}{m^2} + \ldots)$$

> It is thus the case that in the first 4^m whole numbers, there must be one which is not divisible by any of the first m members.

[13] For the detailed proof see WA2, p. 325. Significantly, Wittgenstein's proof consists solely of arithmetical transformations. As this proof has already been commented on cogently in Mancousu and Marion (2003), it is not discussed further here.

[14] Mancousu and Marion (2003) have pointed out the historical and systematic relationships between Behmann's and Wittgenstein's transformations of Euler's proof. Behmann's remarks, however, are unpublished.

[15] This is not recognized by Mancousu and Marion (2003).

Most significantly, Wittgenstein never refers to equation (5) in his proof. He only refers to the inequality[16]

$$\sum_{i=1}^{i=4^m} \frac{1}{i} > \prod_{i=1}^{i=m} \frac{1}{1-\frac{1}{i+1}} - 1 \tag{8}$$

The right-hand side of this inequality is identical to

$$\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \ldots \cdot \frac{m+1}{m} - 1 = \prod_{i=1}^{i=m} \frac{i+1}{i} - 1, \tag{9}$$

which in turn is identical to m as can easily be seen by canceling out the fractions. Thus, in order to derive (8) Wittgenstein simply substitutes $\prod\limits_{i=1}^{i=m} \frac{1}{1-\frac{1}{i+1}} - 1$ for m in the proven inequality $\sum\limits_{i=1}^{i=4^m} \frac{1}{i} > m$. Thanks to (8) he has now defined two *finite* series of natural numbers: one going from 1 to 4^m, and one going from 1 to m. Because both series are finite, one can now always find a member of the series going from 1 to 4^m that is not divisible by a member of the series going from 1 to m in a finite number of steps. Thereby Wittgenstein does not refer to a series of primes but to a series of natural numbers. (8) does not refer to the primes $p_1 \ldots p_m$ but to the natural numbers. If no natural number x, $x \leq m$, can divide a given number y, $y \leq 4^m$, then no prime $p_1 \ldots p_n$, $p_1 \ldots p_n \leq m$ will divide y either. By identifying a number z, $m < z \leq 4^m$, that is also not divisible by any number x, $x \leq m$, one can find the new prime p_{n+1}, $m < p_{n+1} \leq 4^m$, by factoring z. Wittgenstein's proof thus does provide a method of constructing new primes, by identifying the finite intervals that contain them and applying DR-PRIMES. Wittgenstein's proof satisfies his conception of mathematical proofs, and does not involve any circularity as it simply abstains from the use of $p_1 \ldots p_n$ in (8).

Unlike Wittgenstein, Behmann does not base his proof upon (8). Instead, he refers to the invalid equation (5) by substituting the right-hand side of (5) for the real number r in $\sum\limits_{i=1}^{2^{2r-1}} \frac{1}{i} > r$. He thus derives the following inequality, abbreviating $\prod\limits_{i=1}^{n} \frac{p_i}{p_i-1}$ with $\prod_n$:

$$\sum_{i=1}^{2^{2\cdot\prod_n -1}} \frac{1}{i} > \prod\nolimits_n \tag{10}$$

[16] The simpler inequality $\sum\limits_{i=1}^{i=4^m} \frac{1}{i} > \prod\limits_{i=1}^{i=m} \frac{1}{1-\frac{1}{i}}$ is not valid because of the case i=1.

Unlike the series $\prod_{i=1}^{i=m} \frac{1}{1-\frac{1}{i+1}} - 1$ used in (8), $\prod_{i=1}^{n} \frac{p_i}{p_i - 1}$ (or $\prod_n$) is not necessarily a natural number m. That is why Behmann uses the variable r in his proof instead. Inequality (10) can be used to inductively define a series of primes starting with 2, again by defining the interval that contains a new prime p_{n+1} ($p_{n+1} > p_1 \dots p_n$) and applying DRPRIMES. In this respect the relationship between (10) and (8) is similar to the relationship between (2), referring to prime numbers (cf. p. 16), and (3), referring to natural numbers (cf. p. 17).

In comparison to Behmann's transformation, Wittgenstein's transformation of Euler's proof has several advantages. (i) It does not refer to the invalid equation (5). (ii) Like the original inductive definition (1) (cf. p. 15), an application of (10) presumes the computation of all preceding prime numbers. Inequality (8), however, can be applied directly to any natural number m. One can thus also answer a question about the prime exceeding some natural number m without computing a long series of primes. (iii) Wittgenstein's proof does not exceed the realm of natural numbers, whereas Behmann has to refer to the realm of real numbers. (iv) Inequality (8) shows that the infinity of primes can be proven without referring to an arbitrary number of primes $p_1 \dots p_n$ in the proof.

Finally, Behmann's transformation of Euler's proof was part of a project to define a general procedure which transforms indirect, non-constructive proofs into direct, constructive proofs. In so doing he devises a strategy to transform logical deductive implications, and thus presumes that logical formalizations of the arithmetical propositions are available (cf. Mancousu and Marion (2003), p.176–178). In fact, as Mancousu and Marion (2003) (p. 178) point out, Behmann is unable to base his transformation of Euler's proof on this general strategy. As in the case of Wittgenstein's proof, Behmann's transformation essentially depends on a refinement of the statement that the harmonic series diverges. Any effort to define a general logical strategy of transforming indirect, non-constructive proofs in any case runs up against Wittgenstein's objection to logical formalizations of mathematical propositions and proofs. From Wittgenstein's point of view, an adequate transformation of Euler's proof depends only on an *analysis* of the left-hand side of (5). No general logical strategy can serve, only a concrete and adequate formalization of singular mathematical propositions. The transformation of Euler's proof follows this approach to the letter.

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