A MODEST VERSION OF WITTGENSTEINIAN FINITISM

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Abstract. This paper explains the critique of the Church-Turing theorem against the background of a modest version of finitism inspired by Wittgenstein.

Contents

1. Introduction

This paper explains my critique of the Church-Turing theorem against the general background of several significant distinctions drawn by Wittgenstein. According to my interpretation, these distinctions present an alternative understanding of the relation between computation and logic to that implied in Church's or Turing's proof. Against this conceptual background, undecidability proofs of FOL are ruled out. Instead, the decidability of FOL constitutes a natural conjecture that logicians should address.

The New Logic Project was originally motivated by my efforts (i) to make his programmatic claims concerning decidability reasonable and (ii) to present a systematic account of his conception of a "new logic" that he placed in opposition to the mathematical logic established by Frege and Russell. However, I do not pretend to offer any exegesis of Wittgenstein in section 2. Instead, I intend to point out some distinctions and views espoused by Wittgenstein that challenge the Church-Turing theorem and that inspired me to work out a decision procedure for FOL. What is at issue is not the interpretation of Wittgenstein but rather the reasoning for reconsidering the Church-Turing theorem. Similarly, by advocating a modest version of Wittgensteinian finitism in section 5, I refer to neither any discussion about Wittgenstein's standpoint on the foundations of mathematics in general nor his relation to finitism in particular,[1] nor do I pretend to do justice to Wittgenstein's way of discussing mathematics and its foundations, which did not aim to provide a systematic account. I also do not make any effort to relate the "modest version of Wittgensteinian finitism" that I advocate to other versions of finitism. My intention is merely to sketch the general framework that has motivated my work on the decision problem.

2. Wittgenstein's Critique of Undecidability Proofs

The critique of Church's and Turing's proofs is inspired by Wittgenstein. According to my understanding of Wittgenstein's remarks on the foundations of mathematics in general and undecidability proofs in particular, Wittgenstein advocated a computational view of mathematics and logic and criticized undecidability proofs as underdetermined.

Unfortunately, Wittgenstein related his critique to Gödel's proof of the incompleteness of axiomatic systems of arithmetic (cf., in particular, RFM I, Appendix I and Part V, §§18f.) and not directly to Turing's or Church's proof. However, he did engage with the decidability of FOL in the first place (see section 4 below), and his critique concerns not the subtleties of the undecidability proofs but rather the principal question of whether such proofs are compelling. My critique of Church's and Turing's proofs is inspired by Wittgenstein's critique of Gödel's proof. However, in contrast to Wittgenstein, I claim that his kind of critique applies primarily to undecidability proofs of FOL. More precisely, it concerns the expressibility of a hypothetical decision function for FOL-provability by means of a propositional Σ_1 function in L_A, not Gödel's way of expressing unprovability by means of a Π_1 formula on the basis of expressing "y is a PA proof of x". Gödel's proof does not consider expressing provability as a computable function (cf. Def. 46 in [**?**], p. 171, where Gödel defines 'x is a provable formula' in terms of a notion 'of which we cannot assert that it is recursive'). His proof is based on the much weaker assumption of recursively defining 'y is a proof of x' and expressing this recursive function as an open formula that he abbreviates as yBx. Provability (in PM or PA) is merely expressed by binding the free variable y in yBx. Gödel's proof derives a contradiction through diagonalizing $\exists y yBx$ and assuming it or its negation to be provable within an axiomatic system such as PM or PA. This contradiction can also be used to show that these axiomatic systems are incomplete (given their (ω-)consistency). Such an option is not available in the case of undecidability proofs of FOL since FOL can be conceptualized in an axiom-free form.

In relation to Gödel's proof, Wittgenstein stated that instead of reducing the assumption of decidability to absurdity, one could similarly "withdraw the interpretation" of G (cf. RFM I, Appendix I § 8 and § 10).[2] However, he did not distinguish between Gödel's Π_1 formula G and Σ_1 formulas that are intended to express recursive functions. Instead, his critique concerns the principal point that undecidability proofs leave open the possibility of rejecting the reliability of an intended interpretation in the diagonal case instead of reducing the assumption of decidability to absurdity. Wittgenstein emphasized that undecidability

[1]Frascolla, Marion and Rodych were the first to relate Wittgenstein's philosophy of mathematics to finitism, and I have profited considerably from their thorough work, e.g., from [Frascolla (1994)], [Marion (1998)] and [Rodych (1999)].

[2]The exact understanding of this remark and the question of whether it is based on a profound understanding of Gödel's proof are extensively debated. I have discussed Wittgenstein's critique of Gödel's proof as well as the literature on this critique in detail in [**?**]. Unfortunately, I did not argue in that paper that Wittgenstein's critique should more correctly be related to undecidability proofs of FOL than to Gödel's proof.

proofs are underdetermined, indirect proofs and that whether one is inclined to give up the search for a decision procedure due to such a proof depends on what one accepts as a criterion for proof (RFM I, Appendix I, §14f., and V, §22). Unfortunately, Wittgenstein argued very generally and did not relate his critique to specific assumptions of undecidability proofs that are, in fact, affected by his critique. The previous section intended to overcome this deficiency by relating Wittgenstein's critique to the *expressing theorem* in Church's proof and *Lemma 2* in Turing's proof. I consider this the most promising way to extract systematic value from Wittgenstein's critique.

However, much more important than the particular understanding of Wittgenstein's admittedly vague remarks on undecidability proofs is the fact that some of his key distinctions relating to mathematical and logical proofs suggest a serious alternative to the axiomatic method of mathematical logic. The following three sections sketch this alternative. The specific critique of Church's and Turing's proofs in the last section and the motivation to explicate a decision procedure for FOL is best understood against this general background.

3. Wittgenstein's Distinctions: Language of Computation vs. Language of Logic

In his *Tractatus logico-philosophicus* from 1918, before the theory of computation was established, Wittgenstein drew several key distinctions, such as those between *operations and functions* (TLP 5.25; cf. TLP 5.2-5.3), between *formal and material concepts or properties/relations* (TLP 4.122-4.1274) and between *"showing" and "saying"* (TLP 4.121-4.1213, 4.126[3]). He accused Frege's and Russell's conception of mathematical logic (the "old logic", in Wittgenstein's words, as opposed to his conception of a "new logic") of not drawing these distinctions (TLP 4.122[3], 4.126[2], 4.1272[7], 5.25[3]; cf. PT 5.005341, CL letter 68 from 19.8.1919, p. 124).

According to Wittgenstein, basic arithmetic "functions", such as the successor, addition, multiplication, subtraction, and division functions, as well as basic so-called logical "truth functions", such as negation, conjunction, disjunction, implication, and quantification, are all *operations*. Operations can be (i) iteratively applied (whereas functions cannot, according to Wittgenstein's terminology; cf. TLP 5.251); (ii) reversed; (iii) the number of times they are applied can be counted; and (iv) their application can be limited in number or conditioned by the identification of some formal property. The application of operations starts from some input, and the applied operations compute an output. Arithmetic operations and logical operations form different internal related systems that manipulate different types of "forms": arithmetic operations generate numbers, whereas logical operations generate logical forms. Such forms are not referred to by symbols but rather are identified by their outer form and the syntactic rules applied to them.

Arithmetic properties (such as being a natural number), *arithmetic relations* (such as equality), *logical properties* (such as provability), and *logical relations* (such as logical equivalence) are all *formal* properties or relations. *Material* properties/relations are stated in the form of propositional functions, while *formal* properties/relations "are not expressed by means of a function" (TLP 4.126) but rather are identified by applying operations.

According to Wittgenstein, it cannot be "*said*" but rather is "*shown*" that formal properties or relations hold. One way to interpret this distinction is by claiming that formal properties are decidable by means of purely symbolic manipulation, while material properties are stated by interpreting propositional functions within a logical formalism. One cannot identify that material properties hold by means of purely syntactic operations that concern only the syntactic forms of symbols. Instead, material properties refer to objects; whether those objects have certain (material) properties is determined in accordance with the interpretation of symbols and can be proven only through axioms. By contrast, formal properties are not properties of objects that are referred to by symbols but rather are properties of the forms of symbols. The forms of symbols are identified by equivalence transformations within a formalism, e.g., by transformations of equalities in primitive arithmetic or by normal form transformations in logic. Formal properties are identified via a mechanical procedure that reduces members of equivalence classes to ideal representatives (TLP 6.113, 6.1203, 6.122, 6.126, 6.1265; cf. [Lampert (2017b)] and [Lampert (2018b)] for details).

Although Wittgenstein's conception of computation in terms of the application of operations shares certain basic intuitions with Church's or Turing's specification of computability, it resists the representation of computability within a logical formalism. Computability is conceptualized in the form of purely syntactic manipulations of symbols, independent of any interpretation of those symbols and prior to any axiomatization. The distinction between computation and stating true or false propositions is a difference of the kind of language used, according to this analysis. This distinction of the language of computation from the language of logic means that the former language is autonomous and that any attempt to express formal properties such as provability with a language based on FOL and to capture those properties by means of axiomatic systems based on FOL is secondary.

The solvability of formal problems depends on the chosen language; as a prominent example, one might consider the language of algebra, which makes it possible to decide the possibility of geometric constructions.[3] In the case of undecidability proofs of axiomatic systems such as Q, there is no need to blame *the undecidability* of the underlying FOL (cf. [Smith (2013)], p. 302). Instead, it is the syntax and semantics of FOL that are to blame for the inability to express the decidability of FOL within FOL itself. Translating a decision procedure for FOL-provability into a propositional function that can be diagonalized via Gödelization results in intended interpretations that are not compatible with the syntax and semantics of FOL. From this point of view, FOL is the wrong language with which to adequately formalize a decision procedure for FOL. This argument questions not the usual syntax and semantics of FOL but rather the adequacy of their application to formalize a presumed decision procedure for FOL within a language based on FOL.

Computation as carried out by means of computer programs, Turing machines or recursive functions is based on the application of operations, in Wittgenstein's terminology. *Applying* operations is categorically different to *stating* or proving the truth of propositions. The syntax of computation is, roughly speaking, a syntax of recursive rules, not of propositional functions. While logical reasoning and, more specifically, automated theorem proving in FOL may well be computable, this activity is not fully expressible within the language of FOL, according to Wittgenstein's understanding of computation. If one does not assume

[3] [Kvasz(2008)] reconstructs the history of mathematics as a history of the development of the languages of mathematics that, step by step, made it possible to increase the power that can be achieved in expressing and solving problems. He offers numerous further examples that show the dependence of the expressibility and solvability of mathematical problems on the mathematical language used. Unfortunately, he does not consider the question of whether the Church-Turing theorem should alternatively be interpreted as showing the limits of the language of logic (FOL) with respect to expressing its own decidability, rather than stating that FOL is undecidable.

but rather doubts that a decision procedure defined in the language of computation (be it ordinary code or code reduced to the language defined in terms of recursive functions or Turing machines) is adequately expressed in the syntax and semantics of propositional functions and captured in Q (or any consistent extension of Q), then one will most likely interpret undecidability proofs of FOL as reducing the assumption that languages based on FOL can be used to express a decision function for FOL, instead of the decidability of FOL, to absurdity.

Whereas tradition has it that the concept of a function is a general one, of which computable functions are a specification, Wittgenstein advocated for distinguishing the realm of the computable from the realm of propositions by syntactic means: Computable properties can be identified by defining operations, which are capable of iterated application (or "self"-application). However, such properties cannot be expressed by propositional functions since their intended interpretation in terms of self-reference is not reliable (cf. TLP 3.33-3.333). Given these distinctions, the endeavour to express and capture FOL-provability by means of a propositional function is based on a misconception; it is a kind of categorical mistake that is relevant as soon as diagonalization is of interest. This critique questions neither the informal concept of computation nor its explication according to Church's or Turing's thesis; rather, it questions the formal presentation of what is computable within the language of logic and the consequences thereof.

According to this critique, the decidability of FOL cannot be measured with respect to the possibility of expressing a decision procedure by means of a characteristic propositional function in some language based on FOL. As long as the decision problem is not solved positively, no hypothetical reasoning can settle the matter. From Wittgenstein's point of view, the decision problem is the problem of defining an algorithmic equivalence procedure within FOL that reduces FOL formulas to ideal symbols that enable one to decide formal properties such as provability or contradiction. Given the rationale for his critique, no meta-logical reasoning can rule out the possibility of a decision procedure that does not reach beyond FOL.

4. Wittgenstein's Conjecture

Based on his distinction between the realm of computation (what can be shown) and the realm of propositions (what can be said), Wittgenstein conjectured that FOL is decidable in a letter to Russell from 1913. Before he was confronted with the undecidability results of Gödel and Turing, he had already formed a principal and programmatic conception of computation and logic, although he was not interested in working out the technical details. In the late thirties, he was in close contact with Turing, and he was one of the first to whom Turing sent his paper from 1936/7. I have previously argued that he never abandoned his conjecture even after he was confronted with Turing's proof (cf. [Lampert (2019)]).

However, Wittgenstein was neither interested in spelling out his programmatic conjecture nor interested in technically elaborating his basic distinctions, nor did he spell out his critique of undecidability proofs in detail. He seemed to simply be satisfied with his general conviction that the decidability of FOL cannot be measured by considering the (im)possibility of expressing the formal property of FOL-provability by means of a propositional function within a language based on FOL. This rather programmatic rejection of any negative solution to the decision problem is not satisfying. However, rejecting Wittgenstein's conjecture is not convincing, either, as long as this rejection simply takes the common acceptance of the Church-Turing theorem as its standard.[4] If one takes Wittgenstein's point of view seriously, then the Church-Turing theorem cannot be presumed when evaluating his conjecture. One should instead evaluate this challenge by evaluating attempts to demonstrate Wittgenstein's conjecture.

Discussions of foundational issues often end in spelling out various conceptualizations, aims, methods and standards that are difficult to evaluate without prejudice. Thus, prior to such a discussion, it is expedient to focus on the Church-Turing theorem as a decisive point of divergence on the shared background of the syntax and semantics of FOL. Since mathematical logic is a well-established discipline, a serious debate on its foundations must start from identifying some definite anomaly in any case.

This is the reason why I considered the first task in discussing Wittgenstein's rather programmatic view to be to attempt to elaborate a decision procedure for FOL. I started from the basic conviction that pure equivalence transformation within FOL is the sole foundation on which the algorithm should be based. This conviction is in line with Wittgenstein's conception of a logical proof. According to his conception of a new logic, a logical proof is not a derivation of theorems from axioms. Instead, more generally, a logical proof decides logical properties (such as provability or contradiction) by means of nothing other than the mechanical transformation of FOL formulas into ideal representatives of equivalence classes. These ideal representatives provide the criteria for identifying the logical properties of the formulas they represent. A positive solution to the decision problem reduces both provable (refutable) formulas and non-provable (non-refutable) formulas to expressions that allow one to identify the logical properties in question from the purely syntactic properties of the resulting expressions.

The FOL-Decider is the result of this work. Thus, this critique of the Church-Turing theorem can be either verified or falsified by checking this procedure and the proof of its logical foundations.

5. A Modest Version of Wittgensteinian Finitism

Rejecting the expressibility of a decision function for FOL within a language based on FOL (instead of rejecting the assumption of the decidability of FOL) is an essential feature of what I call "a modest version of a Wittgensteinian finitism". It is *modest* in the sense that it reduces Wittgenstein's rather general critique of mathematical logic and the axiomatic method to a specific critique of the Church-Turing theorem. It does not question the consistency of Q or consistent extensions of Q, nor does it question Gödel's incompleteness proof of PA. It also does not reject Church's or Turing's thesis, nor axiomatic or diagonal proof methods in general. The semantics and syntax of FOL, Q and PA are accepted, and meta-mathematical investigations are appreciated. All that is called for is a cautious application of the *expressing theorem* when diagonal cases are involved. Likewise, the diagonal method is unproblematic (or almost trivial) as long as no intended interpretations of diagonalized propositional functions are involved that are considered as the hypothetical results of a translation procedure for a decision procedure. Thus, neither Cantor's theorem nor the unsolvability of the halting problem is questioned, for example.

Instead, it is the Church-Turing theorem and its underlying assumption of the unrestricted expressibility of computable functions within the language of logic that are rejected.

[4] Cf., among others, [Floyd (2005)], p. 95; [Potter (2009)], p. 181-183; and [Landini (2007)], p. 112-118.

The elaboration of a decision procedure refers only to well-known and accepted proof methods, inference rules and notions of computation. Thus, this kind of critique is also *modest* in the sense that it applies only modest proof methods that make up the core of everyday mathematics and logic. It can be called *Wittgensteinian* since both the rejection of the general expressibility of computable functions within a language based on FOL and the conviction that FOL is decidable are motivated by ideas and distinctions presented by Wittgenstein.

The conception advocated for here can be called a version of *finitism* because it calls for a cautious use of generalizations. Generalization to an infinite number of cases presumes a certain similarity among the cases. However, this similarity is not a given when indifferently quantifying over "all interpretations", without distinguishing reliable and unreliable interpretations of FOL or L_A formulas. There is no basis for grounding the general claim of the *expressing theorem* on induction or some other sort of generalization from a finite number of well-defined cases when intended interpretations of diagonal cases are involved. The undecidability proofs of FOL and Q are based on strong, general claims that should apply even to abnormal diagonal cases that are only hypothetically envisaged in indirect proofs. It is apparent that a proof is lacking for the strong claims stated in the *expressing theorem* or in *Turing's Lemma*, which hold only if these claims apply to the hypothetically assumed diagonal cases that induce contradictions.

Again, this critique is not based on a general rejection of any basic concepts or proof methods used in mathematics. It rejects neither the concept of (potential or even actual) infinity in general nor quantification over infinite domains in particular. Likewise, neither the diagonal method in general nor the diagonalization of propositional functions and the Diagonalization Lemma in particular is rejected here. Instead, all that is argued for is a cautious use of generalizations by showing how one can go astray in the case of the reasoning involved in undecidability proofs. However, being cautious in generalizing claims also means that one should not infer from this critique of the undecidability proof of FOL that the concepts and methods involved must be rejected from the outset. In this respect, the argument is, once more, *modest*.

Furthermore, a Wittgensteinian *finitism* is a version of *finitism* in the sense that it advocates for a computational conception of logic and mathematics *for its own sake* that cannot be threatened by limiting theorems proven by the axiomatic method. Solving mathematical and logical problems by means of finite computations is its ultimate aim since it "recognizes the uncertainty of all speculations" (Kronecker, cf. [Meschowski (1967)], p. 238, my translation).

In the following, I will characterize this version of finitism more generally as an alternative to axiomatic mathematics, an alternative for which I believe Wittgenstein advocated.

To more precisely specify this version of finitism, two forms of mathematics are to be distinguished by their proof methods and underlying languages: axiomatic mathematics and algorithmic (or computational) mathematics. Axiomatic mathematics was established, roughly speaking, by the developments in mathematics from 1850 to 1950. Frege and Russell established the language of logic as the universal formal language of all reasoning and, in particular, of mathematical reasoning. The invention of logical formalism went hand in hand with the development of set theory as a new ontology in which the language of logic is interpreted. This emerged as a new framework for formalizing informal mathematical reasoning. This framework gave rise both to new problems (such as questions of decidability and the limits of computation) and to new proof methods for solving those problems. Logical formalization, axiomatization and diagonalization were all established within this framework and used as new proof methods for investigating the limits of computation.

Algorithmic mathematics, in contrast, can be seen as a continuation of "old-fashioned", classical mathematics, seeking to solve mathematical problems by inventing new algorithms based on specific mathematical languages. It does not attempt to formally express and capture informal, ordinary reasoning through the use of a universal language; instead, it is oriented towards solving specific problems arising from the application of mathematics in science. While axiomatic mathematics interprets FOL as a universal language and aims for generality, algorithmic mathematics aims for computability by inventing specific algorithms designed to solve specific problems. Its language is the language of computation – program code, not FOL. The decision problem of FOL is merely one computational problem, in addition to others, that one should attempt to solve through computational logic by looking for specific logical transformations into ideal symbols in ideal logical notations. Algorithmic mathematics does not rely on meaningful propositions that state or describe some non-symbolic realm, nor does it deduce theorems from axioms; it instead solves formal problems, identifies formal properties and relations, or constructs formal entities (such as numbers, sequences or geometric objects) through the pure manipulation of symbols. Wittgenstein's philosophy of mathematics can be read as a defence of the aims of classical, core mathematics in the face of the emergence of a new kind of higher mathematics that extends beyond what is computable.

While Wittgenstein's remarks ultimately express that he, at least to some extent, rejects the use of the language of logic in mathematics, he nevertheless pretends not to intend to revise mathematics. A modest version of Wittgensteinian finitism does not question the axiomatic method but does interpret its outcome as placing limits not on core, computable mathematics but rather on axiomatic mathematics pursued within the language of logic. The incompleteness of PA, first and foremost, says something about the limits of arithmetic when formulated in L_A (which is based on FOL) and axiomatized by PA; it does not say anything about attempts to pursue arithmetic by using other (non-logical) languages and other (non-logical) rules. According to an algorithmic point of view, FOL and arithmetic are separated by their languages. Quantification in a logical formalism is categorically distinguished from existence claims proven by constructive methods and from generality claims proven by induction in mathematics. Algebraic equations are interpreted not as propositional functions containing free variables that can be bound by logical quantifiers but rather as part of a specific algebraic language that calls for its own methods of computing solutions for the "unknowns". Arithmetic equality is a formal relation relating to numbers and distinguished from logical identity in terms of a propositional function referring to objects. Arithmetic and logical operations are not subsumed under a general concept of functions but rather (i) are separated as operations from an extensional, set-theoretic concept of functions and (ii) are separated from each other by constituting different sorts of languages.

The core of mathematics is computation, and this is an autonomous practice that cannot be limited by the axiomatic method. This method goes beyond the proof methods of computational mathematics and establishes its own limits. These limits are induced by the use of the language of logic, which is not suitable for expressing decision procedures or solving mathematical problems by computation in general. It is the intention to express formal properties by means of a logical syntax that necessitates to start from unproven axioms that can never fully capture what one intends to express within a language based on FOL.

A modest version of Wittgensteinian finitism (i) separates axiomatic mathematics from computational mathematics, (ii) interprets the limiting theorems of axiomatic mathematics as theorems concerning the framework of the axiomatic method, (iii) urges only cautious use of generalizations based on semantic intuitions and therefore questions the general validity of the *expressing theorem*, (iv) aims to approach mathematics and logic from a purely algorithmic point of view for its own sake, and (v) proves the merits of this approach by means of working out decision procedures based on modest proof methods, for example, a decision procedure for FOL on the basis of nothing but well-known rules of FOL.

Thus, a modest version of Wittgensteinian finitism does not question axiomatic mathematics as long as it does not pretend to place any limits on computational logic or mathematics. The Church-Turing theorem, however, does not merely state that the provability of FOL formulas cannot be expressed within a language based on FOL; instead, it claims that no computer program can be written to decide FOL. The FOL-Decider is designed to refute this claim, thereby demonstrating that finitism, in terms of computational logic and mathematics, is able to achieve results that are impossible to achieve according to or within axiomatic mathematics.

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