Iconic Logic and Ideal Diagrams: The Wittgensteinian Approach

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Abstract. This paper provides a programmatic overview of a conception of iconic logic from a Wittgensteinian point of view (WIL for short). The crucial differences between WIL and a standard version of symbolic logic (SSL) are identified and discussed. WIL differs from other versions of logic in that in WIL, logical forms are identified by means of so-called ideal diagrams. A logical proof consists of an equivalence transformation of formulas into ideal diagrams, from which logical forms can be read off directly. Logical forms specify properties that identify sets of models (conditions of truth) and sets of counter-models (conditions of falsehood). In this way, WIL allows the sets of models and counter-models to be described by finite means. Against this background, the question of the decidability of first-order-logic (FOL) is revisited. In the last section, WIL is contrasted with Peirce's iconic logic (PIL).

1 Introduction

This paper outlines an alternative to standard symbolic logic (SSL), namely, Wittgenstein's iconic logic (WIL), as a basis for first-order logic (FOL), while avoiding the algorithmic details.[1]

I call the outlined approach "Wittgensteinian" for two reasons: (i) it is inspired by Wittgenstein's early philosophy of logic, and (ii) I wish to distinguish it from Peirce's conception of an iconic logic (PIL). However, I will not present any justification demonstrating that the outlined conception of logic is indeed that of Wittgenstein's early works, nor will I compare the details of Wittgenstein's and Peirce's approaches. Instead, I will focus on the programmatic ideas and fundamental concepts of this Wittgensteinian approach to iconic logic (WIL). In doing so, I intend (i) to make manifest that FOL can be pursued within different paradigms and (ii) to encourage others to work within a Wittgensteinian paradigm of iconic logic.

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[1] Algorithms that realize some of Wittgenstein's ideas concerning logical proofs are available at the following link:
http://www2.cms.hu-berlin.de/newlogic/webMathematica/Logic/home.jsp.

I begin by given the rationale behind WIL (section 2). In the main body of the paper, I explain the conception of proof in WIL and the crucial notion of ideal diagrams as representations of logical forms (sections 3 to 5). I then allude to several significant differences that arise when applying WIL and SSL by addressing the questions of adequate formalization and decidability (sections 6 and 7). Finally, I distinguish WIL from PIL (section 8).

Since the concepts of *logical forms* and *ideal diagrams* are crucial, I will define them here at the outset. Concrete examples and explanations of the concepts used in these definitions will be given below in sections 2 to 5.

Logical Form: The logical form of a first-order formula ϕ is the form of the conditions for truth and falsehood that hold for all formulas that are logically equivalent to ϕ.

According to WIL, ideal diagrams represent logical forms unambiguously. Ideal diagrams are unique representations of equivalence classes of logical formulas. I define them by using (i) a pole-group notation that Wittgenstein introduced in his early writings and (ii) minimal disjunctive normal forms of first-order logic (minimal FOLDNFs). The complete details will be presented in sections 4 and 5.

Ideal Diagram: An ideal diagram is the translation of the set of minimal FOLDNFs that is generated from an initial formula ϕ into Wittgenstein's pole-group notation.

Paraphrases of ideal diagrams, in turn, are the results of a mechanical reading algorithm for ideal diagrams. They make use of a standardized informal language that makes explicit how ideal diagrams should be read as representations of the conditions for the truth and falsehood of instances of initial formulas.

2 The Case for the WIL Approach

Russell writes the following in [Russell(1992)], p. xvi:

> The fundamental characteristic of logic, obviously, is that which is indicated when we say that logical propositions are true in virtue of their form. [...] I confess, however, that I am unable to give any clear account of what is meant by saying that a proposition is "true in virtue of its form".

In SSL, "logical propositions" are defined as formulas that are true in all interpretations. In this sense, SSL places priority on semantics. Accordingly, it does not make sense to characterize logical propositions as "true in virtue of their form". The set of logical propositions is defined not by any specific logical form shared by all logical propositions but rather by the characteristic of being true in any interpretation. In the case of FOL, this means that logical propositions cannot be identified algorithmically by evaluating single interpretations because the number of possible interpretations is infinite.

By contrast, WIL can be characterized as a logic that is intended to fulfill Russell's desideratum. In general, the primary aim of WIL is to assign logical forms to equivalence classes of logical formulas. It is important to recognize that such a conception is reasonable only if one does not rely on either paraphrases or interpretations based on the structure of the *logical formulas* whose logical properties are in question. Such methods of reading or evaluating formulas do not refer to anything that is common to all formulas in the same set of logically equivalent formulas and that may thus serve to identify conditions for the truth or falsehood of propositions sharing the same logical form.

According to [Etchemendy(1999)], there are two ways of understanding the semantics of a formal language. In the *representational view*, different models and counter-models represent different *logically possible* configurations of the world. According to this view, "interpretations" are understood as conditions for the truth value of a sentence. Instances of propositional function variables are fixed, and their meanings do not change with varying interpretations; only their truth values do. By contrast, in the *interpretational view*, different models and counter-models correspond to the assignment of different actual extensions to expressions. This conception does not consider "logical possibilities" or "meaning" in terms of conditions for truth and falsehood. The interpretational view is the standard view of mathematical logic, for example, in Tarski's semantics. The representational view, by contrast, is commonly adopted in philosophical approaches to the semantics of FOL. WIL essentially adopts this view; hence, referring to models and counter-models is equivalent to referring to conditions for truth and falsehood in terms of various logically possible states of the world. According to WIL, the general task of logic is to distinguish conditions for truth and falsehood within a space of *logical possibilities* by identifying the logical form of admissible instances of logical formulas.

In WIL, the logical form of a formula must first be revealed, and it is not until such a logical form has been identified that one can answer the question of what such a form contributes to the representation of conditions for truth and falsehood. As in the case of ordinary propositions, the outer form of a logical formula disguises its logical form. This is so for the following reasons:

1. Any set of logically equivalent formulas is infinite, and although all of the equivalent formulas in such a set share the same logical form, they may have different outer forms. For example, although formulas such as P, $P \vee P$, $P \vee Q \wedge \neg Q$ and $P \vee \neg(R \vee \neg R)$ differ from each other, instances of these different formulas share the same conditions for truth and falsehood.
2. Consequently, one cannot paraphrase an arbitrary logical formula such that
   (a) the paraphrase clarifies what each sign contributes to the representation of the conditions for truth and falsehood (i.e., how each part of the formula specifies certain properties of models or counter-models),
   (b) the signs are *unambiguously* paraphrased to achieve such a clarification (i.e., identical signs are paraphrased identically and different signs are paraphrased differently), and
   (c) all of the (finite number of) *non-redundant* paraphrases of the conditions for truth and falsehood are provided (i.e., all paraphrases that do not

contain any part that can be eliminated without resulting in a paraphrase of a different set of models or counter-models).

By contrast, in WIL, one and only one ideal diagram is assigned to all equivalent formulas, and a proper reading algorithm for such ideal diagrams satisfies conditions 2(a) to 2(c). In doing so, such an algorithm "reads off" the logical form from an ideal diagram.

WIL qualifies as an "iconic logic" because ideal diagrams identify logical forms by their syntactic properties. The features of ideal diagrams serve as identity criteria for sets of (counter-)models that share certain properties. Syntax is prior to semantics in WIL in the sense that for a given formula, the properties of models and counter-models are identified prior to and independently of the evaluation of that formula with respect to single interpretations.

According to WIL, not only the outer form of ordinary language but also the outer form of logical formulas can lead to (logical, linguistic or philosophical) misunderstandings. WIL avoids such misunderstandings by revealing logical forms through equivalence transformation. Such a procedure elucidates our implicit understanding of the construction of logical formulas and what it contributes to specifying conditions for the truth and falsehood of propositions.

3 Logical Proofs

In SSL, logical proofs derive theorems from axioms (or auxiliary assumptions) within a correct and complete calculus. In WIL, however, a proof procedure transforms initial logical formulas into ideal diagrams that enable the identification of the corresponding logical form. Hence, logical proofs in WIL are not merely proofs of logical theorems. A proof in WIL answers the more general question of how an initial formula contributes to identifying conditions for truth and falsehood in general. The proof of a logical theorem (or, likewise, a logical contradiction) is merely a special case of this general procedure.

Because ideal diagrams identify conditions for truth and falsehood and, consequently, also allow one to decide whether the initial formulas are "true in all interpretations", a proof procedure in WIL amounts to a decision procedure. I will discuss the general question of decidability in section 7. For now, it may suffice to say that the crucial challenge in WIL is to specify algorithms for transforming logical formulas into ideal diagrams. In the remainder of this paper, I will present a programmatic overview of WIL, without discussing the technical details of the algorithms for solving this problem. In the following two sections, however, I will address the question of how to specify the ideal diagrams that result from the aforementioned transformation from initial formulas.

4 Ideal Diagrams I - Propositional Logic

From 1912 to 1914, Wittgenstein developed his so-called *ab*-notation as a means of uniquely representing conditions for the truth and falsehood of propositions of a certain logical form.[2] He illustrated this notation with various diagrams of several logical formulas. He used similar diagrams in [Wittgenstein(1994)], remark 6.1203, to demonstrate how to identify tautologies by applying syntactic criteria to the resulting expressions. Instead of the *ab*-notation, he used T and F as "poles" representing the possibilities of truth and falsehood. Wittgenstein also suggested transforming his diagrams into a simpler pole-group notation that corresponds to certain disjunctive normal forms (DNFs) (cf. [Wittgenstein(1979)], p. 102, and [Wittgenstein(1997)], letter 30). His notation was intended to apply not only to propositional logic but also to FOL (cf. [Wittgenstein(1979)], p. 95f). In a letter to Russell, he even conjectured that applying his notation to FOL would enable the identification of tautologies throughout the entire realm of FOL (cf. [Wittgenstein(1997)], letter 30). However, he never spelled out in detail how to apply his notation to arbitrary FOL formulas, nor did he discuss in detail how to achieve unique representations of logical forms in propositional logic (or even FOL). The following is an attempt to revisit Wittgenstein's claim and specify in more detail what is needed in order to represent logical forms by means of ideal diagrams. In this short paper, I cannot elaborate all of the rules for generating such diagrams from logical formulas. Instead, I will focus only on their general properties.

I will initially restrict the discussion to propositional logic. In this case, the application of the well-known Quine-McCluskey algorithm to obtain a set of minimal DNFs is a crucial step in the generation of ideal diagrams. Minimal DNFs distinguish sufficient conditions for truth (the disjuncts) and non-redundant parts of those conditions (the conjuncts); cf. condition 2(a) on p. 3. This allows conditions for the truth of admissible instances of an initial formula to be read off. The same applies to conditions for falsehood, if one also generates the set of minimal DNFs of the negation of the initial formula. By the nature of minimal DNFs, no part of the paraphrase of any single minimal DNF is redundant; cf. condition 2(c) on p. 3.

However, the minimal DNFs of a formula of propositional logic are not unique. Therefore, their paraphrase does not satisfy condition 1 on p. 3. For example, formula (1) has the two minimal DNFs expressed in (2) and (3):

$$P \wedge \neg Q \vee \neg P \wedge Q \vee P \wedge R \vee Q \wedge R \tag{1}$$

$$P \wedge \neg Q \vee \neg P \wedge Q \vee P \wedge R \tag{2}$$

$$P \wedge \neg Q \vee \neg P \wedge Q \vee Q \wedge R \tag{3}$$

However, if one regards a representation of the entire finite set of minimal DNFs as the ideal diagram, then the requirement of uniqueness is satisfied. One might object that if both formulas (2) and (3) together are taken to be part of the ideal diagram, then the non-redundancy requirement for the paraphrases of ideal diagrams (cf. condition 2(c) on p. 3) is not satisfied. However, I propose to interpret

[2] Cf. his letters to Russell during this period, reproduced in [Wittgenstein(1997)], as well as Wittgenstein's *Notes on Logic* and his *Notes dictated to G.E. Moore*, both printed in [Wittgenstein(1979)].

the non-uniqueness of the minimal DNFs in terms of an "ambiguity of the logical form". This ambiguity is represented by a corresponding ambiguity within the ideal diagram. Therefore, the ideal diagram must represent all alternative minimal DNFs, and thus, no alternative is superfluous. Each alternative might be called a representation of a "variant" of the logical form. Such an alternative must not, in itself, contain any redundancy in the description of the conditions for truth or falsehood. However, all alternatives in the entire set of such alternatives must be considered in order to characterize the ambiguity of the logical form. The extent of that ambiguity can be quantified by the number of minimal DNFs. This is why condition 2(c) on p. 3 refers to "*all* non-redundant paraphrases". The non-redundancy requirement applies only to each paraphrase individually and not to the set of all such paraphrases.

Instead of representing all minimal DNFs as a set, one may distinguish components that are common to all minimal DNFs from those that are different by using a two-dimensional notation as in expression (4):

$$P \wedge \neg Q \vee \neg P \wedge Q \vee {}^{P \wedge R}_{Q \wedge R} \tag{4}$$

In contrast to (4), only one minimal DNF is generated from ¬(1) to represent the conditions for the falsehood of propositions that are admissible instances of (1):

$$\neg P \wedge \neg Q \vee P \wedge Q \wedge \neg R \tag{5}$$

Wittgenstein's *ab*- or TF-notation also translates logical constants. T- and F-poles assigned to atomic propositions indicate the affirmation and negation, respectively, of the corresponding atomic propositions. Single T-pole-groups (F-pole-groups) list the non-redundant subconditions that constitute a sufficient condition for truth (falsehood). These single pole-groups, in turn, are arranged into lists of the sufficient conditions for truth or falsehood. Making use of the features of this pole-group notation, one obtains the following ideal diagram as a representation of the logical form of (1):

$$\mathtt{T} - \left\{ \begin{array}{c} \{\mathtt{T} - P,\ \mathtt{F} - Q\}, \\ \{\mathtt{F} - P,\ \mathtt{T} - Q\}, \\ \left\{ {}^{\mathtt{T}-P,\ \mathtt{T}-R}_{\mathtt{T}-Q,\ \mathtt{T}-R} \right\} \end{array} \right\}$$

$$\mathtt{F} - \left\{ \begin{array}{c} \{\mathtt{F} - P, F - Q\}, \\ \{\mathtt{T} - P, \mathtt{T} - Q, \mathtt{F} - R\} \end{array} \right\}$$

Fig. 1. Ideal diagram of (1)

From this diagram, it is possible to directly read off the conditions for the truth and falsehood of admissible instances of formula (1). The following is a paraphrase of this ideal diagram:[3]

[3] I abstain here from cumbersome references to instances of atomic formulas. Thus, I refer to P instead of "an admissible instance of P", etc. I also abstain from specifying the trivial algorithm for paraphrasing ideal diagrams of propositional logic.

An admissible instance of formula (1) is true iff

- P is true and Q is false, or
- P is false and Q is true, or
- one of the following alternatives:
  P is true and R is true / Q is true and R is true.

An admissible instance of formula (1) is false iff

- P is false and Q is false, or
- P is true and Q is true and R is false.

This paraphrase of the ideal diagram is valid for all formulas equivalent to (1). Unlike the paraphrases of propositional formulas in general, the paraphrase of the ideal diagram of a propositional formula identifies common features of the models and counter-models for all formulas in the set of logically equivalent formulas. Instead of specifying *single* interpretations as models and counter-models, as is the case in model theory, the ideal diagram *describes* the properties of *sets* of models and counter-models. This difference is significant when there are an infinite number of models and counter-models, as in FOL.

In the case of logical theorems, Wittgenstein's TF-notation makes explicit that the conditions for truth and falsehood do not depend on the truth values of any atomic propositions. $P \vee \neg P$ and $Q \vee \neg Q \vee (R \wedge S)$, for example, are logically equivalent theorems. As long as one is interpreting logical formulas, the interpretation of the first formula seems to depend on the truth values of instances of P, whereas the interpretation of the second seems to depend on the truth values of instances of P, Q and R. According to WIL, however, this is an illusion caused by the outer forms of the formulas. As soon as one is relating semantics not to initial formulas but rather to ideal diagrams, it becomes clear that logical theorems and their instances do not depend on the truth values of any atomic propositions. They all have the same conditions for truth and falsehood; they all say the same thing, namely, nothing. This becomes apparent upon the application of a reduction algorithm that deletes atomic propositions in the process of generating the ideal diagram. One may use $T - \{\square\}$ to represent that the conditions for truth comprise the entire space of logical possibilities, whereas $F - \{\blacksquare\}$ (or, alternatively, $F - \{\}$) may be used to represent that the conditions for falsehood are not included within the space of what is logically possible. This is the shared logical form of all "logical propositions" that Russell was unable to present within his symbolism (cf. p. 2).

By applying the well-known Quine-McCluskey reduction algorithm in propositional logic and several rather trivial rules for generating ideal diagrams within Wittgenstein's TF-notation, the concept of proof in the WIL version of propositional logic is fully defined. From this, it is clear what must be achieved within FOL: one must find a procedure for generating minimal DNFs in FOL (= FOLDNFs) and translate the resulting sets of minimal FOLDNFs into ideal diagrams in Wittgenstein's notation. [Lampert(2017b)] prescribes how to achieve this for the fragment of FOL that starts from formulas that do not contain any dyadic sentential connectives in the scope of quantifiers. [Lampert(2017c)] generalizes this prescription to a decision procedure for the FOL fragment that is translatable into disjunctions of conjunctions of formulas that do not contain $\vee$ in the scope of quantifiers. [Lampert(2017a)] defines an effective procedure for generating FOLDNFs in general and specifies an effective procedure for minimizing them.[4] However, this procedure does not fully satisfy the requirements for a proof procedure of WIL because it does not fully minimize the FOLDNFs in every case. The task of finding such a procedure remains an open problem. In the following section, I first define the syntactic properties of minimal FOLDNFs and then specify (i) how to translate them into ideal diagrams and (ii) how to paraphrase those ideal diagrams. This discussion should clarify the meaning of a representation of a logical form in FOL, although to date, no general algorithm has been specified that can generate such representations in all cases.

5 Ideal Diagrams II - First-Order Logic

Minimal FOLDNFs are defined in terms of primary formulas, which correspond to negated and non-negated atomic formulas in the DNFs of propositional logic. The term negation normal forms (NNFs) refers to formulas that contain $\neg$ only directly to the left of atomic propositional functions and $\wedge$ and $\vee$ only as dyadic connectives.

Primary Formula:

1. An NNF that does not contain $\wedge$ or $\vee$ is a primary formula.
2. NNFs that contain $\wedge$ or $\vee$ are primary formulas iff they satisfy the following conditions:
   (a) Any conjunction of n conjuncts ($n > 1$) is preceded by a sequence of existential quantifiers of minimal length 1, and all n conjuncts contain each variable of the existential quantifiers of that sequence.
   (b) Any disjunction of n disjuncts ($n > 1$) is preceded by a sequence of universal quantifiers of minimal length 1, and all n disjuncts contain each variable of the universal quantifiers of that sequence.
3. Only NNFs that satisfy condition 1 or 2 are primary formulas.

Primary formulas represent the limit to which quantifiers can be driven inwards by applying PN laws, i.e., the equivalence laws that are used to generate prenex normal forms if applied in the opposite direction. Cases 2(a) and 2(b) above are the only cases in which PN laws cannot be applied to drive quantifiers any farther inwards. As will be shown below, primary formulas can be translated into diagrams within Wittgenstein's notation that satisfy the conditions for ideal representations given that no conjunct or disjunct is redundant. [Lampert(2017a)], section 2, specifies an effective algorithm for generating

[4] This procedure as well as others are implemented at and can be applied via the link given in footnote 1.

FOLDNFs (i.e., disjunctions of conjunctions of primary formulas).[5] Thus, there is no difficulty in establishing this part of the procedure for generating ideal diagrams for FOL.

Minimal FOLDNFs are defined as follows:

Minimal FOLDNF: A minimal FOLDNF is a disjunction of conjunctions of primary formulas that satisfies the following condition: If any number of conjuncts or disjuncts (whether they occur inside the scope of quantifiers, i.e., within the primary formulas, or outside the scope of quantifiers) is deleted, then the resulting formula is not equivalent to the initial one.

Thus, no conjunct or disjunct is redundant in the case of minimal FOLDNFs. Defining a general procedure for generating the set of *minimal* FOLDNFs from FOLDNFs is the problematic part of implementing Wittgenstein's concept of proof within FOL.

The crucial difference between FOLDNFs and the DNFs of propositional logic is the use of primary formulas. In order to clarify how they contribute to specifying the properties of models and counter-models, I will describe how they can be translated into ideal diagrams in Wittgenstein's notation and then illustrate how to paraphrase those ideal diagrams. Consider first an example of a minimal primary formula to motivate its translation into some other notation:

$$\exists y(\forall x(\neg Fxx \vee Hxy) \wedge Gy) \tag{6}$$

Suppose that this primary formula is part of a minimal FOLDNF that is equivalent to an initial formula (the details of which are unimportant here). (6) does not satisfy the standards of WIL in two respects: (i) it is equivalent to all formulas obtained by renaming the variables, and (ii) it contains $\wedge$ and $\vee$, although these signs do not contribute to specifying truth conditions in the same way that they do when they occur outside the scope of quantifiers within FOLDNFs. Because (i) is true, the particular type of each variable does not contribute to representing the properties of models (or counter-models). Instead, it is the relations between bound variables and the positions at which those variables occur in the atomic propositional functions that represent the properties of models. Because (ii) is true, $\wedge$ and $\vee$ cannot be paraphrased in the same way both inside and outside the scope of quantifiers in FOLDNFs. Within primary formulas, $\wedge$ does not separate non-redundant subconditions of a sufficient condition for truth, and $\vee$ does not separate sufficient conditions for truth. (i) highlights how conditions 1 and 2(a), as listed on p. 3, contribute to the ability of the outer forms of logical formulas to disguise their logical forms, whereas (ii) highlights the contribution of conditions 2(a) and 2(b).

These problems can be solved by applying the following algorithm to translate primary formulas into their corresponding ideal two-dimensional "primary diagrams" in a Wittgensteinian notation:

[5] FOLDNFs are far less complex than Hintikka's distribute normal forms of FOL; cf. [Lampert(2017a)] for details.

1. Translate the propositional functions. Replace variables with numbers to indicate positions, and denote affirmation by `T` and negation by `F`.
2. Symbolize the relations between the bound variables and their positions. Use forks to connect the numbers of the positions as follows:
   (a) Open forks, e.g., $\langle$, connect the numbers of positions connected by a disjunction and bound by a universal quantifier.
   (b) Closed forks, e.g., $<$, connect the numbers corresponding to all other positions of one and the same variable.

   When a bound variable occurs only once within only one propositional function, no fork is needed.

Proceeding from inside to outside, this algorithm results in the following translation of (6):

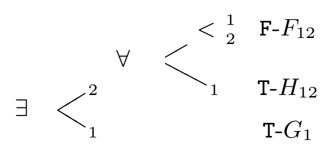

Fig. 2. Ideal diagram of (6)

Thus, the bound variables, $\vee$ and $\wedge$ are eliminated in favour of forks connecting the positions of the corresponding propositional functions.

This ideal diagram can now be paraphrased using a simple procedure that proceeds from outside to inside:

- Some object, the same in the second position of H_{12} and in the first position of G_1, combined with all objects distributed among (i) the first and second positions of F_{12} (where the same object appears in both positions) and (ii) the first position of H_{12}, makes the dyadic propositional function F_{12} false, the dyadic propositional function H_{12} true, and the monadic propositional function G_1 true.

This paraphrase clarifies how the properties of figure 2 identify the properties of models. Open forks represent distributions of objects, whereas closed forks indicate identical objects in different positions.

Here, I omit the cumbersome but trivial specification of (i) the general algorithm for translating FOLDNFs into ideal diagrams of FOL and (ii) the reading algorithm for ideal diagrams of FOL. It should be clear that the crucial problem encountered in an attempt to detail the proof procedure for WIL is specifying a procedure for fully minimizing FOLDNFs.

The translation of a minimal FOLDNF results in a finite ideal diagram in the `TF`-pole-group notation (cf. p. 5), in which each finite group describes the properties of a possibly infinite set of models (or counter-models) and each primary ideal diagram describes certain properties that all (counter-)models in such a set share. Instead of referring to an infinite number of models that may, in turn, involve infinite domains and infinite interpretations of atomic propositional functions, finite descriptions of the properties of such infinite sets are provided, independently of and prior to the evaluation of interpretations. The logical form of the models is identified, thus making it superfluous to explicitly refer to infinity. From the perspective of computability, this is crucial and desirable.

6 Application of Logic - The Question of Adequate Formalization

In this section, WIL is applied for the analysis of the logical forms of ordinary propositions. Two examples are presented that illustrate the misleading form thesis with respect to ordinary propositions, and the standard for adequate formalization according to WIL is explained. I assume the logic to be FOL. Therefore, the logical forms are restricted to logical forms that are expressible in terms of first-order formulas.

Example 1 concerns propositions of the following form:

$$\text{All F's of G's are F's of H's.} \tag{7}$$

This form assumes that F, G and H are variables of atomic propositional functions. The following propositions are instances of (7):

$$\text{All children of mothers are children of fathers.} \tag{8}$$

$$\text{All heads of horses are heads of animals.} \tag{9}$$

$$\text{All bets on winning numbers are bets on prime numbers.} \tag{10}$$

The *logical* forms of (8) to (10) must be independent of any specific internal relations between the meanings of the concepts invoked. From a logical standpoint, any possible combinatoric extension of these concepts is logically possible regardless of how strange, or even inconceivable, such a state of affairs would be. Therefore, mothers also being fathers, horses not being animals and headless horses are all *logical possibilities*. However, this does not mean that such strange possibilities correspond to *truth* conditions of their respective sentences. Instead, according to WIL, distinguishing between logical possibilities that make a sentence true and those that falsify it is a question of adequate logical formalization. The logical form shows how the truth conditions of a complex proposition depend on logically possible extensions of its atomic propositional functions.

The following two logical formulas are reasonable candidates for a logical formalization of propositions instantiating (7):[6]

[6] Standard logic textbooks, such as [Copi(1979)], p. 131f., or [Lemmon(1998)], p. 131f., formalize (9) by (11); by contrast, [Wengert(1974)] argues that only (8) should be formalized by (11), whereas (9) should be formalized by (12).

$$\forall x(\exists y(Fxy \land Gy) \to \exists z(Fxz \land Hz)) \tag{11}$$

$$\forall x \forall y((Fxy \land Gy) \to (Fxy \land Hy)) \tag{12}$$

WIL requires ideal diagrams to make explicit the conditions for the truth and falsehood of the formalized propositions in relation to certain atomic propositional functions. According to WIL, it is not logical formulas but ideal diagrams that are judged to be adequate or inadequate as representations of the logical forms of formalized propositions. Consequently, it is possible to assign *unique* logical forms to unambiguous propositions within this framework. There is no need for formalization criteria that call for a similarity between logical and grammatical forms to allow one to choose among logically equivalent formulas (cf., e.g., [Peregrin & Svoboda(2017)], p. 73).

For simplicity, I will write down only the ideal diagrams for the falsehood conditions of (11) and (12) (cf. figures 3 and 4, respectively). The corresponding representations of the truth conditions are symmetrical in this case.

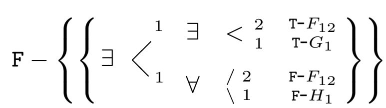

Fig. 3. F-pole-groups of the ideal diagram of (11)

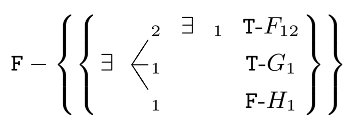

Fig. 4. F-pole-groups of the ideal diagram of (12)

According to an understanding of (8) in which "mother" and "father" refer to "biological mother" and "biological father", respectively, one can view figure 3 as an adequate formalization of the conditions for the falsehood of (8), whereas figure 4 can be seen as an adequate formalization of the conditions for the falsehood of (9). The difference is that (9) is false if there exists a head that is the head of a horse that is not an animal, whereas (8) is not false if there exists a child that is a child of a mother who is not a father. Instead, (8) is false only if some child exists who is a child of a mother but not of a father. In the case of (10), both figures 3 and 4 present reasonable forms for paraphrases of the conditions for falsehood. This situation shows that the meaning of (10) is ambiguous. Overall, this discussion demonstrates that the shared *outer form* (7) does not determine a unique logical form. This, however, does not mean that it is not reasonable to assign a logical form to a certain *ordinary proposition* with respect to a given set of atomic propositional functions. Instead, WIL provides the tools to do so while clarifying the conditions for the truth and falsehood of the initial propositions.

Example 1 has demonstrated that the outer form of a proposition does not determine a unique logical form. The following example illustrates that the outer form of a proposition also does not determine whether a proposition has a proper logical form *at all*. Thus, WIL provides the tools not only to express the conditions for the truth and falsehood of propositions within a logical framework but also to make explicit that certain propositions are not expressible within this framework.

Consider propositions of the following form:

If someone (is in relation) F (to) a G, then a G exists. (13)

The following propositions are instances of (13) (cf. [Montague(1966)], p. 266, and [Quine(1960)], §30):

If someone loves a women, then a woman exists. (14)

If someone seeks a unicorn, then a unicorn exists. (15)

(13) can be translated into the following FOL formula:

$$\exists x \exists y (Fxy \land Gy \to Gy) \tag{16}$$

(16) is a logical theorem. Thus, it seems reasonable to formalize (14) by the ideal diagram of logical theoremhood (i.e., a diagram with empty conditions for falsehood). However, this is not the case for (15), which will most likely be judged to be false. This is why “x seeks y” is commonly regarded as an inadmissible instance of an atomic function variable within FOL.[7] From this it follows that logical forms cannot be assigned to propositions involving such a predicate. Therefore, (15) has no proper logical form, whereas (14) does.

The outer form of a proposition determines neither a unique logical form (cf. (7)) nor whether various propositions of a certain form share a logical form at all (cf. (13)). This is even true in cases of instances of provable formulas. The equivalence of the conditions for truth and falsehood is the criterion for

[7] According to [Quine(1960)], §30, a predicate such as “x seeks y” does not refer to a set of pairs and, thus, does not satisfy the principle of extensionality. However, the question is how one can know this without referring to some failure of logical formalization. For our purposes, it is sufficient to note that mere instantiation of logical formulas does not guarantee that those instances behave in accordance with the laws of logic. Therefore, one must distinguish between admissible and inadmissible instances. According to WIL, instances are inadmissible if they are not judged to be true despite instantiating provable formulas.

adequate formalization when applying logic to propositions, and this cannot be judged without first generating ideal diagrams. According to WIL, interpretation comes last, not first.

7 Application of Logic - The Question of Decidability

A typical argument against Wittgenstein's conception of logic asserts that his understanding of logical proof assumes decidability, which is in conflict with the Church-Turing theorem (cf., among others, [Landini(2007)], p. 118, and [Potter(2009)], p. 181f). However, this argument is not conclusive because the undecidability proof of FOL makes assumptions that WIL rejects.

Turing's undecidability proof relies on a formalization of the code of Turing machines. Furthermore, it relies on a claim that propositions about Turing machines that result from substituting propositional functions for function variables in provable formulas are true. To justify this claim, Turing explicitly refers to the following general principle (cf. [Turing(1936)], p. 262):

> If we substitute any propositional functions for function variables in a provable formula, we obtain a true proposition.

As argued in the previous section, this principle applies only to "admissible instances", i.e., instances that have a certain logical form and, hence, have conditions for truth and falsehood that are expressible within FOL. However, the expressibility within FOL is questionable in the case of diagonalization, which produces self-referential propositions. For example, it is common to reject "This proposition is not true" as an admissible instance of the function variable P in the provable formula $P \leftrightarrow P$. Like other undecidability proofs, Turing's undecidability proof rests on diagonalization. Turing argues that a Turing machine $\mathcal{E}$ that decides on *any* logical formula cannot exist because the decision on a formula involving the formalization of $\mathcal{E}$ in the diagonal case cannot correspond to the behavior of certain machines involving $\mathcal{E}$. The quoted principle does not demonstrate that Turing's formalization of Turing machines is adequate in the diagonal case. Therefore, his inference of the non-existence of a Turing machine $\mathcal{E}$ is a fallacy.[8]

8 Wittgenstein and Peirce

There are many similarities and differences between WIL and PIL.[9] I refer only to the most essential ones in the following.

[8] In fact, I have detailed a decision procedure for pure FOL without identity on the basis of a Wittgensteinian conception of proof (cf. the link given in footnote 1). For the details of a Wittgensteinian critique of undecidability proofs, cf. [Lampert(2017d)].

[9] Cf., in particular, [Shin(2002)] and [Dau(2006)] for detailed elaborations of PIL. [Pietarinen(2006)] provides a game-theoretic interpretation of PIL and relates this interpretation to the later work of Wittgenstein. By contrast, I refer to the early work of Wittgenstein and his conception of a logical proof as a mechanical transformation into ideal diagrams.

Peirce distinguished two purposes of logic: to investigate logical theories and to aid in the drawing of inferences. A logical calculus serves the latter purpose, whereas a logical system serves the former. Such a system should explain what is expressible through logic. To this end, it must not allow for 'any superfluity of symbols' ([Peirce(1931-1958)], 4.373):

> It should be recognized as a defect of a system intended for logical study that it has two ways of expressing the same fact, or any superfluity of symbols, although it would not be a serious defect for a calculus to have two ways of expressing a fact.

Similar to Peirce's distinction between the calculi of symbolic logic and his existential graphs, Wittgenstein drew a distinction between the axiomatic proof method and his own proof method (cf. [Wittgenstein(1979)], p. 109, and [Wittgenstein(1994)], 6.125). On the one hand, he emphasized that the two methods are equivalent (i.e., do not differ in their results; cf. [Wittgenstein(1994)], 6.125f., and [Wittgenstein(1994)], p. 80). On the other hand, he regarded the traditional method of symbolization, which allows for 'a plurality' of equivalent symbols, as defective with regard to the analysis of propositions ([Wittgenstein(1997)], p. 102[3]; see also p. 93[1] and [Wittgenstein(1994)], 5.43):

> If $p = not{-}not{-}p$ etc., this shows that the traditional method of symbolism is wrong, since it allows a plurality of symbols with the same sense; and thence it follows that, in analyzing such propositions, we must not be guided by Russell's method of symbolizing.

Iconic logic can be distinguished from symbolic logic by the search for a procedure for transforming logical formulas into ideal diagrams that do not permit any 'plurality' or 'superfluity' of symbols.

Wittgenstein considered the need for a theory of deduction and for semantics as foundations of pure logic to be a result of a deficient symbolism. He desired to eliminate the need for semantic foundations by identifying 'the sense' of propositions (i.e., the conditions for their truth and falsehood) by means of iconic features of ideal diagrams. According to Wittgenstein, it is not reality (facts) but rather the *logical possibilities* of truth and falsehood that are represented by ideal diagrams. This is why WIL introduces bipolarity as a fundamental property of a proper logical notation, whereas Peirce claims that 'symmetry always involves superfluity' and that symmetries 'are great evils' for 'the purposes of analysis' (cf. [Peirce(1931-1958)], 4.375). In this respect, WIL differs from Peirce's existential graphs, which seem to instead be guided by the desire to represent reality (cf. the above quote from [Peirce(1931-1958)], 4.373, and [Shin(2002)], p. 52). It is for this reason that existential graphs do not correspond to FOLDNFs, nor even to NNFs (given an endoporeutic reading). The question of how to read or interpret existential graphs is a controversial one. From a Wittgensteinian perspective, this very controversy indicates that these graphs share some of the deficiencies of the conventional logical symbolism.

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