Wittgenstein's Conjecture

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Abstract

In two letters to Russell from 1913, Wittgenstein conjectured that first-order logic is decidable. His conjecture was based on his conviction that a decision procedure amounts to an equivalence transformation that converts initial formulas into ideal symbols of a proper notation that provides criteria for deciding the logical properties of the initial formulas. According to Wittgenstein, logical properties are formal properties that are decidable on the basis of pure manipulations of symbols. This understanding of logical properties (such as provability or logical truth/falsehood) is independent of and prior to any interpretation or application of logic. Wittgenstein's conception of logic is incompatible with the undecidability proof of Church and Turing from 1936. Thus, Wittgenstein's conjecture and his understanding of logic appear to be refuted. This paper argues that Wittgenstein did not drew this conclusion and it explains why he never withdrew his conjecture.

1 The Conjecture

In a November 1913 letter to Russell, Wittgenstein conjectured that first-order logic (FOL) is decidable, stating that "there is one Method of proving or disproving all logical prop[osition]s" (CL, p. 54). He illustrated this by means of his *ab*-notation for propositional logic and conjectured that it "must also apply" to the "Theory of app[arent] var[iable]s", i.e., FOL (CL, p. 54). NL, p. 95f., presents the following examples of the simplest quantified expressions:

$$\forall x \varphi x : \quad a - \forall x - a - \varphi x - b - \exists x - b \tag{1}$$

$$\exists x \varphi x : \quad a - \exists x - a - \varphi x - b - \forall x - b \tag{2}$$

Wittgenstein emphasized that his conjecture depended not on the *ab*-notation itself (cf. CL, p. 52) but rather on his conviction that "logical truth" and "logical falsehood" are identifiable by symbolic properties of their representations in a proper notation.

For propositional logic, the well-known method of truth tables may serve as an illustration. While in the "old notation" (NL, p. 93[4]), i.e., "Russell's method of symbolising" (NL 102[3]), tautologies (= logically true formulas) have no common formal property that can serve as their identity criterion, the method of truth tables makes it possible to identify tautologies by the common property that the value "False" does not occur in the column of the main sentential connective.

Wittgenstein, however, envisaged his *ab*-notation, rather than truth tables, as a decision procedure. The main reason for this is that the *ab*-notation is designed to be generalised to the whole realm of FOL. The *ab*-notation does not merely combine truth values of different *types* of propositional functions; instead, it combines all poles of different *tokens* that occur in the initial formula. Thus, it does not presume any internal dependencies of partial expressions. This becomes important as soon as first-order formulas are considered and internal relations become more complex (cf. [Lampert (2017a)] for details). The reference to poles of tokens is the reason why a propositional tautology is identified in the *ab*-notation by the property that its outermost *b*-pole is connected to opposite innermost poles of one and the same propositional variable; cf. figure 1.[1]

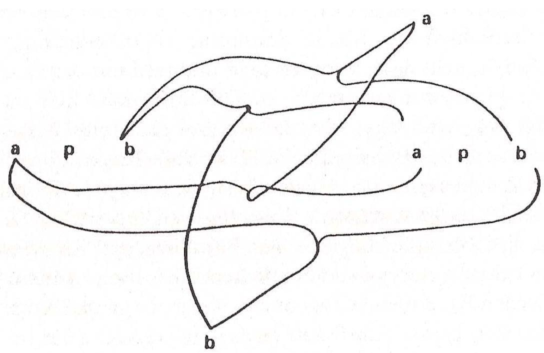

Figure 1: Wittgenstein's *ab*-diagram of $p \equiv p$ (CL, p. 57)

Wittgenstein was unclear on how to realize his understanding of a decision procedure for identity, but he had "NO doubt that it must be possible to find such a notation" (CL, p. 60) for the whole realm of quantified formulas.

Wittgenstein envisaged a method of proving and disproving logical formulas that would differ from both (i) an automated proof search within a calculus of inference rules and (ii) an automated search of models and counter-models with finite domains. In case (i), the search may go on forever if the initial formula is not provable. In case (ii), the search is restricted to finite models or counter-models and, therefore, does not yield a decision for formulas with only infinite

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models or counter-models. Such automated search methods do not support the intuition that FOL is decidable. In fact, modern logic engines do not decide formulas that are neither theorems nor contradictions and have only infinite models or counter-models.[2]

According to Wittgenstein, however, a decision procedure is an equivalence transformation from formulas written in a conventional notation that does not provide criteria for identifying logical properties into ideal symbols of a proper notation that does provide such criteria. Such a method is essentially the same in the cases of both proving and disproving a formula, and it is independent of any semantics or model theory that inevitably faces the problem of infinite domains once quantification becomes involved.

Of course, without being in possession of such a procedure, one can question whether it exists. Nevertheless, given that an automated proof search can be conceived as a reduction to expressions of the form $A \vee \neg A$ or $A \wedge \neg A$ in the case of theorems or contradictions, one might understand the conviction or intuition that it is possible to fully algorithmize an equivalence transformation that also identifies non-theorems or satisfiable formulas by means of syntactic criteria. In addition to truth tables and the *ab*-notation for propositional logic, the translation of propositional formulas into disjunctive normal forms (DNFs) provides another immediate and straightforward illustration of a decision procedure of this kind: an initial formula is contradictory iff each disjunct contains both an atomic formula A and its negation $\neg A$. It is rather simple to extend this kind of decision procedure to fragments of FOL that are known to be decidable; cf. [Lampert (2017a)] for Wittgenstein's *ab*-notation and [Lampert(2017b)] for an equivalence procedure that applies to all FOL formulas that are reducible to FOLDNFs (= DNFs of anti-prenex FOL formulas in negation normal form) that do not contain $\vee$ within the scope of universal quantifiers. In this latter case, a formula is contradictory iff each disjunct contains a "unifiable pair of literals". This, in turn, can be decided by generating subformulas containing only two literals from a disjunct and deciding upon their contradictoriness; cf. [Lampert(2017b)] for details.

Every attempt to generalize decision procedures for fragments of FOL is confronted with the problem that any complete calculus for FOL implies a rule that increases complexity. The problem is to identify criteria for non-provability if complexity increases during the course of an automated proof search. However, this does not necessarily mean that identity criteria are missing in more complex cases. A natural idea for a termination criterion in the case of non-provability in more complex cases that is consistent with Wittgenstein's views on induction in his middle period (cf. section 3, p. 9) is to determine the impossibility of finding a proof along proof paths by detecting loops ("visible recursions") in the proof search given suitable equivalence transformations.

Thus, Wittgenstein's remarks in his letters to Russell have the ingredients of a significant conjecture: (i) a well-defined problem (the "Entscheidungsprob-

[2]Cf., e.g., SYO635+1.p, SYO636+1.p, SYO637+1.p and SYO638+1.p from the TPTP library: http://tptp.cs.miami.edu/∼tptp/cgi-bin/SystemOnTPTP.

lem"); (ii) a method for its solution (equivalence transformation to reduce formulas to ideal symbols); (iii) paradigmatic, partial solutions (the *ab*-notation for propositional logic and fragments of FOL, or, more generally, the reduction to FOLDNFs of fragments of FOL); and (iv) the conviction that partial solutions can be generalized because the decision problem is nothing but the problem of defining an algorithm for symbolic manipulation such that one is left with symbols that provide criteria for deciding the logical properties of the initial formulas.

According to Wittgenstein, a distinctive feature of formal properties, such as properties of mathematical or logical expression, is that they can be represented and identified by symbolic properties of a proper notation. In this respect, they differ from "material properties", which can be represented by propositional functions within a logical symbolism (cf. TLP 4.126). The "confusion" between material and formal properties "pervades the whole of traditional [i.e. old, T.L.] logic" (TLP 4.126) since it treats formal properties as material ones and represents them by propositional functions. The failure to distinguish between material and formal properties is the fundamental mistake of what Wittgenstein calls the "old" logic (4.126), i.e., mathematical logic in the vein of Frege and Russell, as opposed to his "new" logic.

It has often been noted (cf., e.g., the editor's comment in CL, p. 52[3]; [Landini (2007)], p. 112-118[4]; and [Potter (2009)], p. 181-183) that Wittgenstein's conjecture and its implied understanding of logic are refuted by the proofs of Church and Turing from 1936. From the perspective of modern mathematics and mathematical logic, the undecidability proof of FOL is the most obvious and stringent objection to Wittgenstein's understanding of logic and logical proof. In the following, I neither dispute this objection nor argue for or even discuss the systematic value of Wittgenstein's point of view. Instead, I argue that Wittgenstein never relinquished his early conviction that the formal properties of logic and mathematics are not properly representable *within* a logical symbolism. Instead, he acknowledges only symbolic identity criteria of an ideal notation as providing the proper representation of formal properties. This is the basic foundation of Wittgenstein's philosophy of logic and mathematics throughout his life and, as I will show, the reason why he never withdrew his conjecture.

2 The *Tractatus*

Given the significance that Wittgenstein attributed to the decision problem in 1913 and to his *ab*-notation as the framework for solving this problem, it seems surprising that the *Tractatus* does not contain any hint regarding how to decide

[3] "It is [...] interesting that he [Wittgenstein] was looking for a decision method for the whole realm of logical truth. This problem, as we now know, cannot be solved."

[4] "The undecidability of quantification theory is a significant blow to Wittgenstein's conception of logic. [...] it undermines Wittgenstein's hope of finding a notation in which all and only logical equivalents have one and the same representation."

first-order formulas. Instead, Wittgenstein explicitly restricted his method to "cases where no generality-sign occurs" (TLP 6.1203). However, this does not mean that he doubted the feasibility of defining a decision procedure for FOL.

First of all, Wittgenstein was never interested in engaging in the logical business of working out a decision procedure. Furthermore, the intention of the *Tractatus* was not to present the technical details of a decision procedure for logic. Remark 6.1203 was inserted into TS 202 on a separate sheet as late as 1919, and TS 203 and TS 204 do not invoke it. This remark does not extend significantly beyond Wittgenstein's *ab*-notation from 1913. It serves merely to illustrate Wittgenstein's idea regarding the identification of tautologies by means of a decision criterion for a proper notation in the simple case of propositional logic. It is this idea of deciding logical properties based on criteria for a proper notation that is essential to the logic of the *Tractatus*, not the details of its realization in more complex cases. The fact that Wittgenstein restricted the illustration of his method to propositional logic in TLP 6.1203 does not mean that he no longer believed that it could be generalized. He simply did not bother to do so. Moreover, he never maintained that his *ab*-notation is necessary to do this work. TLP 5.1311 and 6.1201 illustrate his idea of identifying logical properties by means of equivalence transformations without alluding to the *ab*-notation, and they relate to formulas of propositional logic as well as to quantified formulas. His conviction is based on the general idea of a decision procedure in the form of an equivalence transformation for converting initial formulas into ideal symbols, not on any specific method of realizing this idea.

Whereas Wittgenstein referred to the logic of the *Principia Mathematica* (PM) in his 1913 letter to Russell, the logic of the *Tractatus* deviates in some respects from the usual understanding of FOL as elaborated in PM. First of all, Wittgenstein rejected the use of identity as a relation between objects in the *Tractatus* (TLP 5.5301). Consequently, he did not permit the use of the identity sign as a primitive symbol in logic (TLP 5.53). Therefore, pure identity statements such as $\forall x\, x = x$ do not fall within the scope of FOL according to the *Tractatus* (TLP 5.534). Nevertheless, logic implies quantifiers and the power to express indefinite quantifiers such as "some" as well as definite quantifiers such as "exactly one" according to the *Tractatus*. Russell showed how to express quantified expressions of this sort within FOL with identity. In doing so, he presumed the usual inclusive reading of quantifiers: $\exists x \exists y(Fx \wedge Fy)$, thus, means "At least one x and at least one (*the same or different*) y such that x is F and y is F", which is equivalent to $\exists x Fx$. Therefore, to express "There are at least two objects that satisfy F", Russell was obliged to introduce identity due to the inclusive reading of quantifiers: $\exists x \exists y(Fx \wedge Fy \wedge x \neq y)$. By contrast, to apply his understanding of logic to identity, Wittgenstein deviated from the usual inclusive reading of bound variables in favour of an exclusive reading that permits the elimination of identity from a proper notation of logic (TLP 5.531ff.). According to his exclusive reading, $\exists x \exists y(Fx \wedge Fy)$ reads "At least one x and at least *one different* y such that x is F and y is F" (= "At least two objects satisfy F"). Thus, $\exists x \exists y(Fx \wedge Fy)$ is not equivalent to $\exists x Fx$ according to the exclusive reading. By his exclusive reading, Wittgenstein wanted to eliminate the need to invoke identity to express quantified concepts such as "at least two" or "exactly one".

This reductionism represents a significant difference with respect to Wittgenstein's attempt to find some proper notation for identity within his *ab*-notation from 1913. Ultimately, his solution to the problem of representing identity in a proper notation of logic was to abandon it in favour of a different reading of quantifiers. Wittgenstein even went one step further in the *Tractatus* by also advocating for a reductive analysis of quantification (TLP 5.52, 6-6.01). In the *Tractatus*, quantified expressions are analysed as truth functions of atomic propositions, which opens the door for problems of infinity in logic. This is a radical difference with respect to his original *ab*-notation, in which quantifiers are not reduced; cf. NL 95f. and formulas (1) and (2) on p. 1. Later, Wittgenstein called his reduction of quantification to propositional logic "his biggest mistake of the *Tractatus*" ([Wright (1982)], p. 152, cf. PG II.8) and returned to a non-reductive analysis of quantified formulas in which quantifiers are accepted as "primitive" (cf. VW, p. 165), as is the case in the *ab*-notation for quantification (cf. NL, p. 95f.).

The logic of the *Tractatus* differs from the usual FOL. The technical details of *Tractarian* logic are still a subject of discussion; cf. [Rogers & Wehmeier (2012)], [Weiss (2017)], and [Lampert / Säbel (2017)]. However, it can be stated that Wittgenstein's reductive analysis of identity and quantification in the *Tractatus* is motivated by his analysis of propositions as truth functions of bipolar atomic propositions. By contrast, his conjecture is based on the general conviction that formal properties such as theoremhood or refutability must be decidable in any proper system of logic because they are inherent properties of the structure of propositions that should be revealed by a proper notation. This conviction is independent of the philosophically motivated analysis of propositions. The analysis of quantified expressions as truth functions of atomic propositions does not imply that a decision procedure must remove quantifiers, nor does an exclusive reading of quantifiers imply that Wittgenstein no longer assumed the usual, inclusive FOL to be decidable. One should distinguish the peculiarities of a *Tractarian* conception of logic from Wittgenstein's general claim that any proper system of logic must enable the identification of logical properties by syntactic criteria during the course of equivalence transformations. This claim applies to the usual calculus of FOL in the vein of Frege and Russell as well as to the specific exclusive and/or reductive *Tractarian* conception of logic.

Wittgenstein's conjecture is independent of the specific inclusive or exclusive reading of quantifiers. This is evident from the fact that he did not consider these peculiarities when discussing the similarities and differences between his proof method and the axiomatic proof method. He called the axiomatic proof conception of Frege and Russell the "old procedure" (MN, p. 109[5]) or the "old conception of logic" (TLP 6.125). This conception relies on "primitive propositions" (TLP 5.43, 6.127f.), i.e., axioms, and "laws of inference" (TLP 5.132) to derive "only tautologies [...] from tautologies" (TLP 6.126, cf. MN, p.109[5]). Wittgenstein accepted the axiomatic method as a method that makes it possible "to give in advance a description of all 'true' logical propositions"

(TLP 6.125). In modern terminology, one might say that he did not question that traditional FOL is correct and complete. Given his critique of identity, one should restrict this claim to FOL without identity. However, what mattered to him in discussing the two different proof conceptions was, first and foremost, that the axiomatic proof method was "not at all essential" (TLP 6.126, cf. TLP 5.132). He denied the ability of such a proof conception to justify the logical truth of the derived tautologies: it neither explains why axioms are tautologies (cf. TLP 6.127f) nor why laws of inference preserve logical truth (TLP 5.131). The axiomatic method requires semantic or extra-logical evidence (cf. 6.1271), which Wittgenstein wanted to make superfluous with his "new" proof conception.

The key idea of this proof conception is to identify the logical properties of propositions by criteria related to their proper notation. A proof in accordance with Wittgenstein's proof conception does not consist of inferring a proposition from other propositions. Instead, it consists of converting a proposition into some equivalent ideal expression that allows one to identify its logical properties. That is why "every proposition is its own proof" (TLP 6.165). This does not merely imply that "all the propositions of logic are of equal status" since "it is not the case that some of them are essentially primitive propositions and others are derived propositions" (TLP 6.127, cf. 5.43). Rather, it additionally means that Wittgenstein did not restrict his proof conception only to "logical propositions" (tautologies). Instead, he referred to "*every* proposition"(see TLP 6.165 above; emphasis mine) and stated "that we can actually do without logical propositions; for in a suitable notation we can in fact recognize the formal properties of the propositions by mere inspection of the proposition themselves" (TLP 6.122).

Since his proof conception applies to "every proposition", i.e., every logical formula, and enables the identification of their logical properties, a proof procedure implies a decision procedure, according to Wittgenstein's proof conception. This can be seen from the following: if the mechanical transformation of an initial formula into its ideal expression in a proper notation does not result in the representative expression for all tautologies, this outcome is sufficient to decide that the initial formula is not a tautology. According to Wittgenstein, the logical *properties* of each logical formula are independent of and prior to its internal *relations* to other propositions (cf. TLP 5.131, 6.12). A proof is a reduction of initial formulas to their ideal representative expressions through equivalence transformations. It is this proof *method*, not the outcome or the specific understanding of quantification and identity, that constitutes the crucial difference between Wittgenstein's "new" logic and the "old logic" of Frege and Russell.

The *Tractatus* does not abandon the conviction that in any proper system of FOL, it must be possible to decide the logical properties of formulas by converting the initial formulas into a proper notation. Hence, the idea that motivated Wittgenstein's conjecture as stated in his letter to Russell 1913 is still prominent in the *Tractatus*.

3 The Middle Period: Before 1936

In his middle period, instead of working out his conception of a "new" logic in more detail, Wittgenstein was much more interested in applying his algorithmic understanding of logic to mathematics.

The idea of applying his understanding of logical proofs in terms of equivalence transformations to mathematics was already Wittgenstein's primary concern regarding his conception of mathematics in the *Tractatus*. He called mathematics "a logical method" (6.2, cf. 6.234). In doing so, he argued not for a logicist reduction of mathematics to logic but for analogous conceptions of proof in both logic and mathematics. Similar to a logical proof, a mathematical proof involves manipulating symbols with the aim of identifying mathematical properties by means of the properties of the resulting ideal expressions. Whereas the propositions of logic are tautologies, the propositions of mathematics are equations (TLP 6.22). In both cases, these propositions are meaningless. Wittgenstein called the method of logic for combining propositions into tautologies the "zero-method" (TLP 6.121); it involves identifying the properties of logical implication and logical equivalence by relating single propositions to senseless tautologies. Likewise, Wittgenstein attributed to mathematics "the method of substitution" (TLP 6.24), which involves combining mathematical terms into meaningless (nonsensical) propositions. However, he emphasized that the application of these methods is not necessary for identifying the properties of logical implication/equivalence or mathematical identity since the internal relations of the related expressions follow from their internal (formal) properties (TLP 6.122, 6.126[4], 6.1265, 6.23-6.2323).

As in logic, "intuition" ("Anschauung", not "self-evidence") is needed in mathematics to solve mathematical problems (TLP 6.233), namely, the intuition that "the process of *calculation* serves to bring about" (TLP 6.2331). This intuition refers to the expressions resulting from logical or mathematical equivalence transformations. Whereas Wittgenstein's *ab*-notation illustrates this intuition in the case of logic, he also conceived of a specific notation for the case of arithmetic. His Ω-notation is designed to support the application of the method of substitution to reduce both sides of an arithmetic equation to identical ideal symbols. TLP 6.241 illustrates this process in the case of the proof of $2 \cdot 2 = 4$, in which both $2 \cdot 2$ and 4 are reduced to $\Omega'\Omega'\Omega'\Omega'x$.[5]

Wittgenstein seamlessly evolved from his *Tractarian* proof conception to the work of his middle period. Since he abandoned the *Tractarian* reductive analysis of quantification in his middle period, he strengthened his claim that the axiomatic proof method of Frege and Russell and his own notation both "achieve the same result" (WVC, p. 92) in logic. He focused not on any difference in the results within FOL but on the irrelevance of the axiomatic method. According to

[5]Cf. the following URL for a full computer program that implements Wittgenstein's Ω-notation for the whole realm of rational numbers:

http://www2.cms.hu-berlin.de/newlogic/webMathematica/Logic/q-decide.jsp

Cf. the "Introduction" for a description of the program.

Wittgenstein, this method is not essential since it is similarly possible to identify tautologies "in my notation" (WVC, p. 92). As in the *Tractatus*, he concluded that tautologies are "indeed quite irrelevant" since his proof method applies to any proposition and is not restricted to inferring theorems from axioms. For this reason, he did not differentiate between proof and decision methods: "That inference is *a priori* means only that syntax decides whether an inference is correct or not. Tautologies are only one way of showing what is syntactical" (WVC, p. 92).

Yet, in his middle period, Wittgenstein was primarily interested in applying his algorithmic proof conception to mathematics. He thus took up the challenge to the *Tractatus* issued by Ramsey, asking how Wittgenstein's "account can be supposed to cover the whole of mathematics" ([Ramsey (1923)], p. 475). In no way did this make him doubt his conviction of the decidability of formal properties. Instead, he began by drawing the same analogies he had drawn in the *Tractatus*: "Logic and mathematics are not *based on* axioms [...] [t]he idea that they are involves the error of treating the intuitiveness, the self-evidence, of the fundamental propositions as a criterion for correctness in logic" (PG, p. 297). Instead, it is intuition ("Anschauung") rather than intuitiveness ("self-evidence") that is relevant; "intuition ["Anschauung"] of symbols" (WVC, p. 219) is not related to a belief in the truth of axioms, as intuitiveness is. What Wittgenstein had said regarding the logical proof of propositions in TLP 6.1265, he then explicitly related to mathematical propositions in PR, p. 192: "the completely analysed mathematical proposition is its own proof". A mathematical proof is "an analysis of the mathematical proposition" (PR, p. 179) rather than a logical derivation from axioms. As in the *Tractatus*, he compared "the method of tautologies" to the "the proof of an equation", stating that both "[make] evident the agreement between two structures".

In Wittgenstein's middle period, he extended his idea of proving mathematical properties by reducing them to symbolic properties of a proper notation to more sophisticated areas of mathematics, such as induction, impossibility proofs and analysis. He rejected any axiomatic or extensional (set-theoretical) understanding of infinity as well as meta-mathematical impossibility proofs in favour of his algorithmic view. In this view, the intent is to reduce any sort of infinite regression, such as in the case of approximating real numbers, to a "visible recursion" in a proper notation (cf. PR, p. 187, 243, and, e.g., PR XIV-XVIII, PG II §32). He positioned this view in opposition to "arithmetic experiments", in which sequences are generated without making manifest the laws governing their construction (PR, p. 235; cf. TLP 6.2331). In contrast, he sought only a progression to infinity in accordance with a "recognizable law" (PR, p. 235). This idea can be traced back to the *Tractarian* concept of operations that generate forms in mathematics or logic through iterative application (TLP 5.25-5.254).

In his middle period, Wittgenstein repeatedly challenged the view that propositions concerning formal properties have any decisive mathematical or logical meaning independent of the possibility of deciding them either by means of a given decision procedure or by inventing one. In contrast to the case of

Hilbert's formalism, Wittgenstein did not understand symbolic manipulation as something that requires additional interpretation. Instead, it is the decision procedure itself that gives meaning to the formal properties and entities in question. In this respect, Wittgenstein advocated for an algorithmic analysis of meaning in pure mathematics and logic. One might object that we do understand statements about logical formulas or mathematical statements without being able to decide them and that much higher work in mathematics goes beyond Wittgenstein's narrow understanding thereof. The essential point of Wittgenstein's view, however, is not the extent of understanding logic or mathematics in the case that no decision procedure is available. Rather, the crucial point is that he adheres to an ideal of a most decisive and precise understanding of logical or mathematical properties that is based on nothing but equivalence transformation within a symbolism and, thus, does not require any reference to entities outside of that pure symbolism. According to Wittgenstein, no higher mathematics or meta-mathematics can threaten this ideal. Instead, mathematics should always strive for the reduction of its concepts and proofs to the realm of decidability. In a case of conflict, this ideal is the standard of proof and cannot be questioned by proof methods that do not adhere to this ideal.

In the *Tractatus*, Wittgenstein was not content to define logical truth on the basis of general validity (TLP 6.1231). In his view, a full understanding of a logical property implies the ability to identify it through symbolic manipulation. He held this view also for mathematical equations, as he explicitly stated in his middle period:

> We cannot *understand* the equation unless we recognize the connection between its two sides.
>
> Undecidability presupposes that there is, so to speak, a subterranean connection between the two sides; that the bridge *cannot* be made with symbols.
>
> A connection between symbols which exists but cannot be represented by symbolic transformations is a thought that cannot be thought. If the connection is there, then it must be possible to see it. (PR, p. 212f.)

It was Wittgenstein's belief that pure mathematics and logic are sciences that concern nothing beyond the finite, rule-guided manipulation of signs, thus ruling out the possibility of undecidability:

> Of course, if mathematics were the natural science of infinite extensions of which we can never have exhaustive knowledge, then a question that was in principle undecidable would certainly be conceivable. (PR, p. 213)

The all-important point is that Wittgenstein's conviction of decidability is rooted in his general understanding of propositions concerning formal properties. This analysis is incompatible with a set theoretical representation of formal properties or any consideration of their decidability within meta-mathematics or any other analysis that goes beyond pure symbolic equivalence transformations.

To understand Wittgenstein's later reaction to undecidability proofs (see the next section), one must consider that his understanding of logic and mathematics is diametrical to the emergence of mathematical logic and the efforts to lay down foundations of mathematics that go beyond what is computable. Before Wittgenstein was ever confronted with Turing's negative solution to the decision problem in 1937, he had already ruled out the possibility of a negative metamathematical answer to the decision problem on the basis of his algorithmic understanding of logic:

> Logic isn't metamathematics either; that is, work within the logical calculus can't bring to light essential truths *about* mathematics. Cf. here the "decision problem" and similar topics in modern mathematical logic. (PG, p. 297)

At the same time, he did not expect any progress to be made in mathematics from the solution to the decision problem since he held that logic and mathematics are analogous in their algorithmic methods but autonomous in their languages, concepts and calculi. It was for that reason that he rejected any logical formalization of mathematics as the basis of the axiomatic method when applied beyond pure logic to mathematics and in undecidability proofs. Whereas many mathematicians feared that a positive solution to the decision problem threatened to make all mathematical questions mechanically solvable without the need for any further human ingenuity, Wittgenstein regarded the decision problem not as a "leading problem" but as a "problem of mathematics like any other" (cf. WA III, p. 268[9], from MS 110, p. 189). There is no evidence that this remark from 1931 was directed against Wittgenstein's early conjecture (against Floyd (2005), p. 95). Wittgenstein questioned not the solvability of the Entscheidungsproblem but rather its importance to mathematics and its foundations. According to Wittgenstein, his conjecture is like any other conjecture in need of an algorithmic solution.

4 Wittgenstein's Reaction to Undecidability Proofs: After 1936/7

Wittgenstein was in close contact with Turing in the late thirties. He was one of the first to read Turing's undecidability proof of 1937.[6] According to Floyd, Turing's proof must have "struck" Wittgenstein ([Floyd (2016)], p. 30). Indeed, this would have been the most reasonable reaction if Wittgenstein had accepted the proof and its underlying method. Alternatively, he could have stuck to his convictions and turned them against the undecidability results of the thirties. One may deny that this is a profound reaction. Yet, given Wittgenstein's convictions and his algorithmic understanding of logic and mathematics,

[6] Cf. Turing's letter to his mother from 11th February 1937, in which he mentions Wittgenstein as the second outside King's College to whom he already had send a copy. Cf. AMT/K/1/54, Turing Digital Archive (http://www.turingarchive.org/browse.php/K/1/54) and [Floyd (2016)], p. 9, footnote 3.

he hardly had any choice unless he was ready to radically change his views and suddenly embrace the application of the axiomatic method in mathematics and meta-mathematics. However, although Wittgenstein changed many of his views throughout his life, there is no evidence that he ever abandoned his critique of the logical formalization of mathematics and meta-mathematics, which characterizes the axiomatic method. This is due to his alternative algorithmic conception of proof that lays at the heart of his philosophy of mathematics throughout his life. Wittgenstein refined this conception in his later philosophy by embedding it into a pragmatic and cultural context that placed focus on a "surveyable representation" ("übersichtliche Darstellung") rather than on mechanical decision procedures. However, this is better understood as a further development in considering the foundations of an algorithmic proof conception, rather than a renunciation of it.

Unfortunately, Wittgenstein did not explicitly discuss Turing's undecidability proof of FOL. Instead, he discussed Gödel's undecidability proof for axiomatized arithmetic theories (such as Peano Arithmetic, PA) in remarks that largely stem from 1937 to 1939.[7] In the following, I will first consider this discussion and then apply it to the decision problem and Turing's proof. I do not deny that there are crucial differences between Turing's and Gödel's undecidability proofs. However, I will argue that these differences do not matter from Wittgenstein's point of view since both similarly question his algorithmic proof conception, and Wittgenstein reacts to both of them with a fundamental critique of the underlying axiomatic method.

4.1 Wittgenstein's reaction to Gödel's undecidability proof

Gödel's undecidability proof proves that there exists at least one formula G in the language of PA (henceforth denoted by L_A) such that neither G nor $\neg G$ is provable from the axioms of PA. If Gödel had proven this result by providing a decision method for provability in PA, this would be in line with Wittgenstein's own proof conception. His paradigm for acceptable, algorithmic proofs of unprovability is manifested in the algebraic proofs of the unsolvability of certain problems within Euclidean geometry, such as the problem of angle trisection with a straightedge and compass (cf. RFM I, appendix I, §14). Such proofs of unprovability are part of a decision procedure that distinguishes between possible and impossible constructions on the basis of their algebraic representations: the angles that can be constructed with a straightedge and compass are those and only those that are representable by algebraic equations that can be solved with nested square roots.[8] This fits with Wittgenstein's algorithmic conception

[7] Wittgenstein delivered his "Lectures on Gödel" (WCL) in the Eastern Term 1938. This lecture begins with his sketch of Gödel's proof from MS 117 (written end of 1937), for a critique of this proof sketch see [Lampert (2006)]. RFM I, appendix I has parallels to WCL. The remarks on Gödel in RFM V, §§18f. are from 1941.

[8] For an implementation of the Kronecker algorithm that allows to decide whether the respective algebraic equations are solvable with nested square roots, cf.

http://www2.cms.hu-berlin.de/newlogic/webMathematica/Logic/k-decide.jsp of proof in terms of a finite transformation of the problem into a representation in some notation that allows one to decide the initial question based on properties of the resulting expressions.

However, Gödel's proof is not of this sort. Instead, it rests on the representation of a formal property, namely, PA-provability, in L_A, i.e., a language that is based on FOL supplemented with constants for numbers and arithmetic functions. This means that provability is expressed by a certain open formula (abbreviated by $\exists yByx$, according to Gödel's definition 46) in L_A iff, for all Gödel numbers n of L_A-propositions, n is provable iff $\exists yBy\overline{n}$ is true according to the intended interpretation of L_A.[9] According to Wittgenstein's proof conception, any intent to represent a formal property, such as provability, by an open formula (propositional function) must be founded on confusion between material and formal properties, which is the fundamental mistake of mathematical logic. In contrast to Gödel, Wittgenstein claimed that formal properties can only be "shown", i.e., identified through a decision procedure; they cannot be "said", i.e., expressed within the formal language to which they apply.

Wittgenstein rejected the application of the axiomatic method in Gödel's undecidability proof of his formula G. He did not do so by referring to the relevant proof of the representability of recursive functions within L_A (cf. theorems V and VII in [Gödel (1931)], p. 186; theorem 13.4 in [Smith (2007)], p. 109; and [Lampert (2018b)] for detailed discussions). Instead, he was aware that he was instead "bypass[ing]" (RFM V, §17, last sentence) Gödel's proof since he was discussing not the details of the proof but rather what could be taken as a "*forcible reason* for giving up the search for a proof" (RFM I, appendix I, §14). For Wittgenstein, this was a question of what counts as a "criterion of (un)provability" (cf. RFM I, appendix I, §14-16, and V, §18f.). According to his algorithmic proof conception, a criterion for a formal property must be a decision criterion in terms of some property of ideal symbols. This is why the proof of the impossibility of trisecting an angle with a straightedge and compass counts as a criterion for giving up the search for such a construction (RFM I, appendix I, §14). By contrast, the criterion for a "forcible reason" to give up the search for a decision procedure is not satisfied by meta-mathematical undecidability proofs since they are based on the representation of a formal property by a propositional function within the formal language itself. According to Wittgenstein, undecidability proofs reduce the possibility to represent provability as a propositional function to absurdity, not the assumption of a decision procedure that is independent of such a representation. Indeed, the verdict regarding the representation of formal properties by propositional functions had lain at the heart of Wittgenstein's critique of mathematical logic since the beginning (cf. TLP 4.126).

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[9] $\overline{n}$ is the expression for the number n in L_A. Under the presumption of the *expressibility* (or definability) of provability, Gödel then proves that the property of provability cannot be *captured* in PA. This means that it is not true for all n that either $\exists yBy\overline{n}$ or its negation is provable from the axioms of PA. Cf. [Smith (2007)], p. 34f., for the definitions of representing (expressing, defining) and capturing properties within PA.

One reason why Wittgenstein thought that formal properties are not representable by propositional functions is that he rejected the possibility of self-referential representations within a formalism based on FOL (cf. TLP 3.332f). He distinguished operations from functions and considered that it is only with operations that self-application comes into play (TLP 5.25f). However, the application of operations is a part of symbolic manipulation and is not something that is expressible by functions within a logical symbolism. Undecidability proofs, meanwhile, rest on diagonalization and, thus, on a formula that is intended to represent that the formula itself does (not) have a certain property. Gödel's formula G, for example, is intended to represent the property of unprovability of the formula G itself. On this basis, he proved that G cannot be captured in PA.[10] This proof method gives priority to semantics (representation) over syntax (capturing). It is only this priority that makes it possible to prove meta-mathematically that an algorithmic proof conception is limited. Such reasoning cannot convince an advocate of the algorithmic proof conception since such an advocate instead places priority on syntax. In the case of conflict, said advocate would deny the definability of the formal property in question. Thus, given G were provable from the axioms of PA, the diagonal case would simply turn out to be such a case of conflict. Therefore, Wittgenstein would not infer that PA is inconsistent but instead would deny that G, in fact, represents its own unprovability (RFM I, appendix I, §8, 10). This is also why Wittgenstein could not accept Gödel's undecidability proof as an proof of incompleteness.

Wittgenstein analysed undecidability proofs as proofs by contradiction (cf. RFM I, appendix I, §14, and cf. PI §125 below). In the case of Gödel's undecidability proof, he mainly considered the contradiction as one between a supposed proof of G and the fact that G represents its own unprovability (RFM I, appendix I, §8, 10, 11). However, his rejection also applies to the so-called syntactic version of Gödel's proof since this version also relies on the assumption that the formal property of provability can be represented within L_A, which involves self-referential interpretations in the diagonal case. No proof of contradiction can be a compelling reason to give up the search for a decision procedure since an advocate of the algorithmic proof conception questions the assumption of representability for the formal property in question.

Wittgenstein compared the contradiction arising in an undecidability proof to a paradox (RFM I, appendix I, §12f, §19). According to Wittgenstein's analysis, so-called semantic paradoxes, such as the Liar paradox, as well as paradoxes of mathematical logic, such as Russell's paradox, rely on the representation of formal properties by propositional functions (cf. TLP 3.33-3.334; WVC, p. 121; and PR, p. 207f.). The problem lies not with the specific properties (semantic

[10] Note that Gödel's syntactic proof presumes the representation of "y is a proof of x", Def. 45, and "x is provable", Def. 46, in [Gödel (1931)], p. 186. Claiming that Gödel's "syntactic" proof does not rely on the representability of arithmetic and meta-mathematical properties (and, in this respect, on semantics) demonstrates a misunderstanding of this proof. Gödel never maintained that. Instead, he made it clear that his syntactic version of the proof is based on the expressibility of provability within L_A; cf. [Gödel (1931)], p. 176. In contrast to Gödel's so-called semantic proof, his so-called syntactic proof presumes only the consistency of PA, not its correctness.

properties vs. set-theoretical properties) but with the analysis of self-reference as something that is expressible by propositional functions and thus capable of being represented in a symbolism based on FOL. The distinction between meta- and object-language is not sufficient to prevent paradoxes, according to Wittgenstein's analysis. Instead, it is the distinction between formal and material properties that must be considered. This distinction comprises both semantic paradoxes and the paradoxes of mathematical logic. It even applies to arithmetic properties and their meta-mathematical correlates. For Wittgenstein, the arithmetic and meta-mathematical interpretations in the language of L_A were not an "absolutely uncontroversial part of mathematics" ([Wang (1987)], p. 49; however, cf. also [Gödel (1931)], p. 149, footnote 14) but rather the outcome of the fundamental mistake of mathematical logic, namely, the assertion that formal properties of mathematics and meta-mathematics can be expressed by propositional functions. Wittgenstein's algorithmic proof conception rules out such a possibility since it maintains that formal properties can be expressed only by symbolic properties of a proper notation. Wittgenstein believed in an algorithmic proof conception as the standard for a rigorous proof that can never be affected by any underlying intended interpretations of a logical symbolism to represent any properties, since such an interpretation necessarily extends beyond the realm of mere symbolic manipulations.

4.2 Wittgenstein's reaction to the undecidability of FOL: PI §125

Turing's undecidability proof for FOL differs from Gödel's proof in many respects. It rests not on recursive functions but on Turing machines; it refers not to L_A and PA but directly to FOL, and it proves without referring to axioms that the property of provability (or, likewise, logical validity) of FOL formulas cannot be decided. However, in terms of Wittgenstein's attitude regarding undecidability proofs, these differences do not matter since Turing's proof similarly rests on the representation of formal properties within the language of FOL. In the case of Turing, these properties are properties of his machines (such as printing a 0 or, in more modern versions of the proof, halting). Thus, Wittgenstein's critique applies to Lemma 2 in [Turing (1936/7)], p. 262. This lemma is proven by referring to the intended interpretations of logical formalizations of the configurations and instructions of Turing machines. In the diagonal case, the provability of a logical formalization of the behaviour of machines involving a decision machine for FOL must be interpreted as a statement about the behaviour of that same machine. However, it is possible to define machines involving FOL such that their behaviour contradicts the intended interpretation. According to Wittgenstein's critique of the method of logical formalization, one is not obliged to infer that no decision machine for FOL exists. Instead, he would reject the interpretation that the provability of the logical formalization is correlated to the behaviour of the formalized machine in the diagonal case. According to Wittgenstein, a logical formalization cannot fully express and capture computation and what is computable. [11]

The fact that Wittgenstein applied his fundamental critique of the axiomatic method and its application within undecidability proofs to the undecidability proofs of FOL is made clear by PI §125. This passage follows up on the remark about "the leading problem of mathematical logic" from 1931 (see above, p. 11) and refers to Ramsey's phrasing of the decision problem. PI §125 is an echo of Wittgenstein's critique of the axiomatic method and its application in meta-mathematical undecidability proofs:

> It is not the business of philosophy to resolve a contradiction by means of a mathematical or logico-mathematical discovery, but to render surveyable the state of mathematics that troubles us - the state of affairs before the contradiction is resolved.
>
> Here the fundamental fact is that we lay down rules, a technique, for playing a game, and that then, when we follow the rules, things don't turned out as we had assumed. So that we are, as it were, entangled in our own rules.
>
> This entanglement in our rules is what we want to understand: that is, to survey.
>
> It throws light on our concept of meaning something. For in those cases, things turn out otherwise than we have meant, foreseen. That is just what we say, for example, when a contradiction appears: "That's not the way I meant it."

Even after Turing's proof, it was obvious to Wittgenstein that the Church-Turing theorem, like Gödel's theorem, rests on "a contradiction that needs to be resolved". This could be done by solving the decision problem. However, this would be a logico-mathematical endeavour. The business of philosophy is instead concerned with the state of mathematics before this discovery. It is to analyse the confusion between formal and material properties that underlies the axiomatic method and its acceptance of meta-mathematical criteria for undecidability. For Wittgenstein, it seemed out of the question that this method should rely on "rules, a technique" that would lead to a situation in which an advocate of the axiomatic method would face the discovery of a decision procedure and then be in need of a philosophical analysis that would explain why "things turn out otherwise" than foreseen.

Wittgenstein inserted §125 from TS 228, written in between 1945 and 1948, into the manuscript of PI, demonstrating that he held onto his conjecture throughout his life. He never accepted a proof method in meta-mathematics that relied on the logical formalization of formal properties. Instead, he adhered to an algorithmic understanding of proof that had already lain at the heart of his early philosophy.

[11] This statement does not in any way question the Turing thesis. Turing's thesis merely states that computability can be reduced to what Turing machines can compute. One can simultaneously accept this and reject the logical formalization of Turing machines. The same applies to the Church thesis: what is questioned is not the reduction of computability to recursive functions but the logical formalization of those recursive functions.

5 Conclusion

From the perspective of modern mathematics and its foundations in mathematical logic, Wittgenstein's point of view might seem like the reactionary dream of an advocate of mathematics in the manner in which it was done before the foundational crisis. To say the least, Wittgenstein's point of view is in stark contrast to the development of modern mathematical logic. Wittgenstein's critical reaction to the emergence of mathematical logic was very general, and he deliberately neither discussed undecidability proofs in detail nor made the effort to solve the decision problem. However, if one wishes to take Wittgenstein's point of view seriously, this work must be done.

Abbreviations of Wittgenstein's works

CL:	Cambridge Letters, Oxford: Blackwell, 1997.
NL:	'Notes on logic', in: Notebooks 1914-1916, Oxford: Blackwell 1979, 93-107.
PR:	Philosophical Remarks, Chicago: Chicago Press, 1975.
PI:	Philosophical Investigations, 4th edition, Indianapolis: Wiley-Blackwell, 2009.
RFM:	Remarks on the Foundations of Mathematics, Massachusetts: M.I.T. Press, 1967.
RPP:	Remarks in the Philosophy of Psychology, Oxford: Basil Blackwell, 1991.
TLP:	Tractatus Logico-philosophicus, London: Routledge, 1994.
VW:	The Voices of Wittgenstein, London: Routledge, 2003.
WAIII:	Wiener Ausgabe. Band 3, Wien: Springer, 1995.
WCL:	'Lectures on Gödel', in: Munz, Ritter (eds), *Wittgenstein's Whewell's Court Lectures*, Oxford: Wiley Blackwell, 2017, 50-57.
WVC:	Wittgenstein and the Vienna Circle, Oxford: Basil Blackwell, 1979.

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